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Approximation in $L_p(\mathbb{R}^d)$ from Spaces Spanned by the Perturbed Integer Translates of a Radial Function

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is only well understood in the "stationary" setting. In this work, we provide lower bounds on the obtainable approximation orders in the "non-stationary" setting under the assumption that Ξ is a small perturbation of $\mathbb{Z}^d$. The functions which we can approximate belong to certain Besov spaces. Our results, which are similar in many respects to the known results for the case $\Xi = \mathbb{Z}^d$, apply specifically to the examples of the Gauss kernel and the generalized multiquadric.

1. INTRODUCTION

Let $C(\mathbb{R}^d)$ denote the collection of all continuous functions $f: \mathbb{R}^d \to \mathbb{C}$ equipped with the topology of uniform convergence on compact sets. For $\phi \in C(\mathbb{R}^d)$, and $\Xi \subset \mathbb{R}^d$, we define $S_0(\phi; \Xi) := \operatorname{span}\{\phi(\cdot - \xi) : \xi \in \Xi\}$, and we let $S(\phi; \Xi)$ denote the closure of $S_0(\phi; \Xi)$ in $C(\mathbb{R}^d)$. The area of radial basis functions has as its motivation the problem of approximating a smooth function $f: \mathbb{R}^d \to \mathbb{C}$ from $S(\phi; \Xi)$ given only the information $f_{|_\Xi}$. The area gets its name from the fact that most of the commonly used functions ϕ are radially symmetric. Three important examples are the polyharmonic spline,

$$\phi(x) := \begin{cases} |x|^{\gamma - d}, & \text{if } \gamma - d \in (0, \infty) \backslash 2\mathbb{N}, \\ |x|^{\gamma - d} \log(|x|), & \text{if } \gamma - d \in 2\mathbb{N}, \end{cases}$$

the Gauss kernel, $\phi(x) := e^{-|x|^2/4}$, and the generalized multiquadric,

$$\phi(x) := \begin{cases} (1 + |x|^2)^{(\gamma_0 - d)/2}, & \text{if } \gamma_0 - d \in (-d, \infty) \backslash 2\mathbb{Z}_+, \\ (1 + |x|^2)^{(\gamma_0 - d)/2} \log(1 + |x|^2), & \text{if } \gamma_0 - d \in 2\mathbb{Z}_+. \end{cases}$$

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Here, $\mathbb{N} := \{1, 2, 3, ...\}$ and $\mathbb{Z}_+ := \{0, 1, 2, ...\}$. The area of radial basis functions encompasses many practical as well as theoretical issues; for a recent survey the reader is referred to [8] (see also [12, 22]). In this paper we are concerned only with the issue of approximation.

Jackson and Buhmann made the simplifying assumption $\Xi = \mathbb{Z}^d$ in their initial investigations (cf. [6, 7, 17]). These initial investigations were followed by others working also under the assumption $\Xi = \mathbb{Z}^d$ (namely, [2, 4, 5, 9, 13, 18, 19, 23]) until the simplified problem was very well understood. In order to describe these results, we need a few more definitions. The space $S(\phi; \Xi)$ can be refined by dilation obtaining

$$S^h(\phi; \Xi) := \{s(\cdot/h): s \in S(\phi; \Xi)\}.$$

Or in other words, $S^h(\phi; \Xi)$ is the closure, in $C(\mathbb{R}^d)$, of the span of the $h\Xi$-translates of $\phi(\cdot/h)$. It is hoped that a smooth function f can be approximated better and better from $S^h(\phi; \Xi)$ as $h \to 0$. In the literature, this is usually quantified by notions of *approximation order*. The essential requirement in the statement "$(S^h(\phi; \Xi))_h$ provides L_p-approximation of order γ" is that

$$\operatorname{dist}(f, S^h(\phi; \Xi); L_p) = O(h^\gamma), \qquad \text{as} \quad h \to 0,$$

for all sufficiently smooth $f \in L_p := L_p(\mathbb{R}^d)$, where

$$\operatorname{dist}(f, A; X) := \inf_{a \in A} \|f - a\|_X.$$

The notion of "sufficiently smooth" should at least include all compactly supported C^∞ functions. We describe now two of the major themes which developed from the above mentioned works. First, if $\hat{\phi}$, the Fourier transform of ϕ, looks like $|\cdot|^{-\gamma}$ near 0, then under various (p-dependent) side conditions it was shown that the ladder $(S^h(\phi; \mathbb{Z}^d))_h$ provides L_p-approximation of order γ, $1 \leqslant p \leqslant \infty$. Typical examples here would be the polyharmonic spline and the generalized multiquadric ($\gamma := \gamma_0$).

The ladder $(S^h(\phi; \Xi))_h$ is known as a *stationary ladder* because it is obtained by dilating the same space $S(\phi; \Xi)$. More generally we may use, as the h-entry of our ladder, the h-dilate of an h-dependent space $S(\phi_h; \Xi)$ to obtain a *non-stationary ladder* $(S^h(\phi_h; \Xi))_h$. It is in this more general setting that the second theme was developed. Starting with a very smooth function ϕ, define $\phi_h := \phi(\kappa(h)\cdot)$ for some function $\kappa: (0, 1] \to (0, \infty)$ which decays to 0 as $h \to 0$. If $\hat{\phi}$ decays exponentially at ∞, then it could sometimes be shown that the non-stationary ladder $(S^h(\phi_h; \mathbb{Z}^d))_h$ provides L_p-approximation of order γ provided that $\kappa(h)$ decays to 0 sufficiently fast with h. Typical examples here are the Gauss kernel and the generalized multiquadric. Although arbitrarily high approximation orders can be obtained if $\kappa(h)$ decays sufficiently fast (see [20, 24, 26] where $\kappa(h) = O(h)$), there is a price to be paid in terms of numerical stability as $\kappa(h)$ decreases. Thus, for practical reasons, it is desirable to know, for a given γ, the slowest decaying κ which still yields L_p-approximation of order γ. For the example of the Gauss kernel, Beatson and Light [2] have shown that if

$$\lim_{h \to 0} \kappa(h)^2 \log(1/h) = \frac{(2\pi)^2}{\gamma},$$

then the non-stationary ladder $(S^h(\phi_h; \mathbb{Z}^d))_h$ almost provides L_∞-approximation of order γ (their error looks like h^γ times some power of $|\log h|$). It is now known (cf. [18, 19]) that $(S^h(\phi_h; \mathbb{Z}^d))_h$ provides L_p-approximation of order (exactly) γ for all $1 \leqslant p \leqslant \infty$ (see also [5] ($p = \infty$), [4] ($p = 2$)).

Recently, there have been a few successful adaptations of some of the abovementioned techniques (i.e., those stationary techniques associated with the first theme) to the more general setting where Ξ is allowed to be scattered throughout $\mathbb{R}^d$. Buhmann *et al.* [10] have shown that if $\hat{\phi} \sim |\cdot|^{-2m}$ near 0, for some $m \in \mathbb{N}$, if certain other side conditions are satisfied, and if Ξ satisfies a mild restriction, then the stationary ladder $(S^h(\phi; \Xi))_h$ almost provides L_∞-approximation of order $2m$ (their error looks like $O(h^{2m}|\log h|)$). Moreover, this approximation is realized by an explicit scheme which, at the h level, uses only the information $f_{|_{h\Xi}}$. The mild restriction on Ξ is that there should exist $C_0 < \infty$ such that every ball of radius C_0 contains an element of Ξ.

Dyn and Ron [14] generalized the results of [10]. They showed that if one has in hand a specific scheme for approximating from the stationary ladder $(S^h(\phi; \mathbb{Z}^d))_h$, then this scheme can be converted into a scheme for approximation from the ladder $(S^h(\phi; \Xi))_h$. Under certain circumstances, it was shown that the latter scheme provides L_∞-approximation of order γ if the former did. Their results apply primarily to functions ϕ for which $\hat{\phi} \sim |\cdot|^{-k}$ near 0 for some $k \geqslant \gamma$. In particular, it was shown that the results of [10] could be obtained by converting the stationary schemes detailed in the paper [13] into the scheme of [10] via a variant of the general conversion method of [14]. Following [14], Buhmann and Ron [11] extended the results of [14] to L_p-approximation for p in the range $1 \leqslant p \leqslant \infty$.

The present work is primarily concerned with providing lower bounds on the L_p-approximation order ($1 \leqslant p \leqslant \infty$) of a given non-stationary ladder $(S^h(\phi_h; \Xi))_h$. Our results begin with the observation that $(S^h(\phi_h; \Xi))_h$ being able to approximate to order $O(h^\gamma)$ the $\mathbb{Z}^d$-translates of a certain very nice function η, in a certain collective sense, implies that $(S^h(\phi_h; \Xi))_h$ provides L_p-approximation of order γ for all $1 \leqslant p \leqslant \infty$ (see the beginning of Section 5). This is reminiscent of the approach taken in [14] where the $\mathbb{Z}^d$-translates of ϕ were approximated from the space $S(\phi; \Xi)$. Due to the niceness of η, the problem of approximating the shifts of η is fairly tractable if Ξ is a sufficiently small perturbation of $\mathbb{Z}^d$, that is, if

$$\boldsymbol{\delta}(\Xi) := \inf\{\delta > 0 : \mathbb{Z}^d \subset \Xi + \delta Q\}$$

is sufficiently small. Here $Q := (-1/2..1/2)^d$ is the open unit cube in $\mathbb{R}^d$. We point out that our ability to approximate the shifts of η from $S^h(\phi_h; \Xi)$ does not require $S(\phi_h; \mathbb{Z}^d)$ to contain any polynomials; this is in stark contrast to the situation in [14] where the ability to approximate the shifts of ϕ from $S(\phi; \Xi)$ is closely related to the polynomials contained in $S(\phi; \mathbb{Z}^d)$. We are subsequently able to identify sufficient conditions which ensure that $(S^h(\phi_h; \Xi))_h$ provides L_p-approximation of order γ for all $1 \leqslant p \leqslant \infty$. These sufficient conditions do not assume the family $(\phi_h)_h$ to be radially symmetric. However, we have made considerable effort in specializing our sufficient conditions to the case where the family $(\phi_h)_h$ is obtained by dilating a fixed radially symmetric function ϕ, namely, $\phi_h := \phi(\kappa(h)\,\cdot\,)$ where κ: $(0, 1] \to (0, \infty)$ is as described above. These specialized results apply in particular to the examples where ϕ is the Gauss kernel or the Generalized Multiquadric. For the Gauss kernel we show that if

$$\limsup_{h \to 0} \kappa(h)^2 \log(1/h) < \frac{\pi^2}{\gamma}, \qquad \text{for some} \quad \gamma \in (0, \infty),$$

and if Ξ is a sufficiently small perturbation of $\mathbb{Z}^d$, then the non-stationary ladder $(S^h(\phi_h; \Xi))_h$ provides L_p-approximation of order γ for all $1 \leqslant p \leqslant \infty$. For the Generalized Multiquadric, we show that if

$$\limsup_{h \to 0} \kappa(h) \log(1/h) < \frac{\pi}{\gamma_1}, \qquad \text{for some} \quad \gamma_1 \in (0, \infty),$$

and if Ξ is a sufficiently small perturbation of $\mathbb{Z}^d$, then the non-stationary ladder $(S^h(\phi_h; \Xi))_h$ provides L_p-approximation of order $\gamma_0 + \gamma_1$ for all $1 \leqslant p \leqslant \infty$.

We have also specialized our general sufficient conditions to the non-stationary scenario where $\phi_h := \phi(h^\theta\,\cdot\,)$ $(0 < \theta \leqslant 1)$ and ϕ is a continuous radially symmetric function satisfying $|\cdot|^{d+1}\phi \in L_1$, $|\hat{\phi}(x)| \sim (1 + |x|)^{-\gamma}$, and $|\lambda^{(k)}(\rho)| = O(\rho^{-\gamma-k})$ as $\rho \to \infty$, $0 \leqslant k \leqslant d+1$, where λ is defined by $\hat{\phi}(x) = \lambda(|x|)$. We show that if $\gamma > d$ and Ξ is a sufficiently small perturbation of $\mathbb{Z}^d$, then the non-stationary ladder $(S^h(\phi_h; \Xi))_h$ provides L_p-approximation of order $\theta\gamma$ for all $1 \leqslant p \leqslant \infty$.

An outline of the sequel is as follows.

In Section 2, we give our precise definition of L_p-approximation order. The results mentioned above, which specialize our general result to the case $\phi_h := \phi(\kappa(h)\,\cdot\,)$ for a fixed radially symmetric function ϕ, are stated in

Section 3 and applied to the examples of polyharmonic splines, the Gauss kernel, and the generalized multiquadric. The proofs of these specialized results are postponed until Sections 6 and 7. Our general results are stated and proved in Section 5 while a number of related technical lemmata are gathered into Section 4.

The following notations are used throughout this work. The natural numbers are denoted by $\mathbb{N} := \{1, 2, 3, ...\}$, while the non-negative integers are denoted by $\mathbb{Z}_+ := \{0, 1, 2, ...\}$. For $x \in \mathbb{R}^d$, we define $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$, while for multi-indices $\alpha \in \mathbb{Z}_+^d$, we define $|\alpha| := |\alpha_1| + \cdots + |\alpha_d|$. The open unit cube and the open unit ball in $\mathbb{R}^d$ are denoted by $Q := (-1/2, 1/2)^d$ and $B := \{x \in \mathbb{R}^d : |x| < 1\}$, respectively. For open $\Omega \subseteq \mathbb{R}^d$, $1 \leqslant p \leqslant \infty$, and $m \in \mathbb{Z}_+$, the Sobolev spaces $W_p^m(\Omega)$ are defined by

$$W_p^m(\Omega) := \left\{ f\colon \|f\|_{W_p^m(\Omega)} := \left(\sum_{|\alpha| \leqslant m} \|D^\alpha f\|_{L_p(\Omega)}^p \right)^{1/p} < \infty \right\},$$

with the usual modification when $p = \infty$. The space of polynomials of total degree at most k is denoted Π_k. The semi-discrete convolution is defined formally by

$$\phi *'_h c := \sum_{j \in \mathbb{Z}^d} c(h_j)\, \phi(\cdot/h - j), \qquad h > 0.$$

For $f \in L_1 := L_1(\mathbb{R}^d)$, we denote its Fourier transform by $\hat{f}(x) := \int_{\mathbb{R}^d} e_{-x}(t) \times f(t)\, dt$, where e_x denotes the complex exponential given by $e_x(t) := e^{ix \cdot t}$. The inverse Fourier transform of f is denoted $f^\vee$. The collection of compactly supported $C^\infty(\mathbb{R}^d)$ functions is denoted by $\mathscr{D}$ and their Fourier transforms by $\hat{\mathscr{D}}$. Moreover, $\mathscr{D}(\Omega)$ denotes the set of all functions in $\mathscr{D}$ whose support is contained in Ω. All derivatives and supports of functions are to be understood as distributional. We employ the convention that 0 times anything is 0; in particular, $0/0 := 0$. We use the symbol const to denote generic constants, always understood to be a real value in the interval $(0, \infty)$ that depends only on its specified arguments. Further, the value of const may change with each occurrence. When using the scaling parameter h, as in $(S^h(\phi_h; \Xi))_h$, it is assumed without further mention that $h \in (0, h_0]$ for some $h_0 \in (0, 1]$. Lastly, we employ the standard notation $\lceil t \rceil$ to denote the least integer which is $\geqslant t$.

2. PRELIMINARIES

In order to make precise the notion, "L_p-approximation of order γ," we need to specify which functions $f \in L_p$ are sufficiently smooth. This will be the Besov space $B_p^{\gamma,1}$ which we now define. Let $\eta \in \hat{\mathscr{D}}$ satisfy $\hat{\eta} = 1$ on a neighborhood of the origin, and for $f \in L_p$, define

$$f_k := \begin{cases} (\hat{\eta}(2\,\cdot\,)\,\hat{f})^{\vee}, & \text{if } \ k=0, \\ ((\hat{\eta}(2^{1-k}\,\cdot\,) - \hat{\eta}(2^{2-k}\,\cdot\,))\,\hat{f})^{\vee}, & \text{if } \ k>0. \end{cases} \tag{2.1}$$

For $1 \leqslant p \leqslant \infty$, $\gamma \geqslant 0$, $1 \leqslant q \leqslant \infty$, the Besov space $B_p^{\gamma,q}$ (see [21]) can be defined as the collection of all tempered distributions f for which

$$\|f\|_{B_p^{\gamma,q}} := \|k \mapsto 2^{\gamma k}\,\|f_k\|_{L_p}\|_{\ell_q(\mathbb{Z}_+)} < \infty.$$

It is known (cf. [21]) that $B_p^{\gamma,q}$ is a Banach space, and as such, is independent of the choice of η (i.e. different choices of η yield equivalent norms). We mention the following continuous embeddings (cf. [21, p. 62]),

$$\begin{aligned} &B_p^{\gamma,q} \hookrightarrow B_p^{\gamma_1,q_1}, && \text{if} \quad \gamma_1 < \gamma \quad \text{or} \ \gamma_1 = \gamma,\ q_1 \geqslant q; \\ &B_p^{k,1} \hookrightarrow W_p^k(\mathbb{R}^d) \hookrightarrow B_p^{k,\infty}, && \text{if} \quad k \in \mathbb{Z}_+; \\ &B_p^{\gamma,1} \hookrightarrow \mathscr{H}_p^{\gamma} \hookrightarrow B_p^{\gamma,\infty}, && \text{if} \quad 1 < p < \infty, \end{aligned}$$

where $\mathscr{H}_p^{\gamma}$ is the potential space normed by

$$\|f\|_{\mathscr{H}_p^{\gamma}} := \|((1+|\cdot|^2)^{\gamma/2}\,\hat{f})^{\vee}\|_{L_p}, \qquad \gamma \geqslant 0, \quad 1 < p < \infty.$$

Incidentally, the function η here is the same as that mentioned in the Introduction.

Definition 2.2. Let $1 \leqslant p \leqslant \infty$, let $\Xi \subset \mathbb{R}^d$, and let $(\phi_h)_{h \in (0\ldots h_0]}$ be a family in $C(\mathbb{R}^d)$. We say that the ladder $(S^h(\phi_h; \Xi))_h$ provides *L_p-approximation of order* $\gamma > 0$ if there exists $c < \infty$ such that

$$\operatorname{dist}(f, S^h(\phi_h; \Xi); L_p) \leqslant ch^{\gamma}\,\|f\|_{B_p^{\gamma,1}}, \qquad \forall h \in (0, h_0], \quad f \in B_p^{\gamma,1}.$$

We mention that it is easy to derive from Definition 2.2 that if $(S^h(\phi_h; \Xi))_h$ provides L_p-approximation of order γ and if $0 < \gamma' < \gamma$, then

$$\operatorname{dist}(f, S^h(\phi_h; \Xi); L_p) \leqslant c'h^{\gamma'}\,\|f\|_{B_p^{\gamma',\infty}}, \qquad \forall h \in (0, h_0], \quad f \in B_p^{\gamma',\infty}.$$

Moreover, if $\gamma' = \gamma$, then the same inequality holds provided we replace $h^{\gamma'}$ with $h^{\gamma}\log(2/h)$.

3. THE RADIALLY SYMMETRIC CASE

Our most general result is Theorem 5.8. There, it is not assumed that the functions $(\phi_h)_{h\in(0,h_0]}$ are radially symmetric. However, the theorem is a bit difficult to read due to its generality. The assumption of radial symmetry turns out to be a convenient means of reducing the complexity of the theorem. In what follows, we assume that the functions ϕ_h are all obtained from a single radially symmetric function ϕ by dilation. The abstract conditions of Theorem 5.8 can then be replaced by other easily verifiable conditions on a certain univariate function related to $\hat{\phi}$. Here are the details:

THEOREM 3.1. *Let $\phi \in C(\mathbb{R}^d)$ be a radially symmetric function with at most polynomial growth at ∞, and assume that $\hat{\phi}$ can be identified on $\mathbb{R}^d\backslash 0$ with $|\cdot|^{-\gamma_0}\lambda(|\cdot|)$ for some $\gamma_0 \in [0,\infty)$ and $\lambda \in C([0,\infty))$ with $\lambda(0)\neq 0$. Define*

$$\bar{\mu} := \sup\{\mu \leqslant \gamma_0 : |\phi(x)| = O(|x|^{\gamma_0-\mu}) \text{ as } |x|\to\infty\};$$

$$m := d + \lceil \gamma_0 - \bar{\mu} \rceil,$$

and assume that,

(i) $|\phi(x)| = o(1)$ *as* $|x|\to\infty$ *if* $\gamma_0 = 0$;

(ii) $\gamma_0 > \lceil \gamma_0 - \bar{\mu}\rceil$ *if* $\gamma_0 > 0$;

(iii) $\lambda \in C^m(0,\infty) \cap C^{d+1}(0,\infty)$;

(iv) $|\lambda^{(k)}(\rho)| = O(\rho^{\varepsilon-k})$ *as* $\rho\to 0$, $\forall 1\leqslant k\leqslant m$;

(v) $|\lambda^{(k)}(\rho)| = O(\rho^{\gamma_0-d-\varepsilon})$ *as* $\rho\to\infty$, $\forall 0\leqslant k\leqslant d+1$,

for some $\varepsilon\in(0,1)$. If Ξ is a sufficiently small perturbation of $\mathbb{Z}^d$, then the stationary ladder $(S^h(\phi;\Xi))_h$ provides L_p-approximation of order γ_0 for all $1\leqslant p\leqslant\infty$. If, in addition to the above, there exists θ, a, $N\in(0,\infty)$ such that

(vi) $\sup_{0<\rho<\infty}(\exp(-a\rho^\theta)/|\lambda(\rho)|) < \infty$;

(vii) $|\lambda^{(k)}(\rho)| = O(\rho^N \exp(-\rho^\theta))$ *as* $\rho\to\infty$, $\forall 0\leqslant k\leqslant d+1$,

and if we define $\phi_h := \phi(\kappa(h)\cdot)$, $h\in(0,1]$, for some $\kappa\colon (0,1]\to(0,\infty)$ satisfying

$$\limsup_{h\to 0} \kappa(h)^\theta \log(1/h) < \frac{\pi^\theta}{\gamma_1}, \qquad \textit{for some} \quad \gamma_1\in(0,\infty),$$

then the non-stationary ladder $(S^h(\phi_h;\Xi))_h$ provides L_p-approximation of order $\gamma_0+\gamma_1$ for all $1\leqslant p\leqslant\infty$ whenever Ξ is a sufficiently small perturbation of $\mathbb{Z}^d$.

In order to demonstrate the utility of Theorem 3.1, we consider now a few examples.

EXAMPLE 3.2. *Polyharmonic Spline.* Let $\gamma > d$ and define $\phi := |\cdot|^{\gamma-d}$ if $\gamma - d \notin 2\mathbb{N}$, or $\phi := |\cdot|^{\gamma-d}\log(|\cdot|)$ if $\gamma - d \in 2\mathbb{N}$. We will show, as an application of Theorem 3.1, that the stationary ladder $(S^h(\phi; \Xi))_h$ provides L_p-approximation of order γ for all $1 \leqslant p \leqslant \infty$ whenever Ξ is a sufficiently small perturbation of $\mathbb{Z}^d$.

According to [16], $\hat{\phi}$ can be identified on $\mathbb{R}^d \backslash 0$ with $\pm \text{const}(d, \gamma)|\cdot|^{-\gamma}$. So, in terms of Theorem 3.1, λ is constant, $\bar{\mu} = d$, and $m = \lceil \gamma \rceil$. It is now trivial to verify that conditions (i)–(v) are satisfied (with $\gamma_0 := \gamma$, $\varepsilon \leqslant \gamma - d$). The desired conclusion now follows from Theorem 3.1.

EXAMPLE 3.3. *Gauss Kernel.* Let $\phi := e^{-|x|^2}/4$, and let κ: $(0, 1] \to (0, \infty)$ satisfy

$$\limsup_{h \to 0} \kappa(h)^2 \log(1/h) < \frac{\pi^2}{\gamma}, \qquad \text{for some} \quad \gamma \in (0, \infty).$$

Define

$$\phi_h(x) := \phi(\kappa(h)\, x) = e^{-\kappa(h)^2\, |x|^2/4}, \qquad x \in \mathbb{R}^d, \quad h \in (0, 1].$$

We will show, as an application of Theorem 3.1, that the non-stationary ladder $(S^h(\phi_h; \Xi))_h$ provides L_p-approximation of order γ for all $1 \leqslant p \leqslant \infty$ whenever Ξ is a sufficiently small perturbation of $\mathbb{Z}^d$.

For that note that $\hat{\phi}(x) = (4\pi)^{d/2}\, e^{-|x|^2}$. Hence we fall into the hypothesis of Theorem 3.1 with $\gamma_0 = \bar{\mu} = 0$, $m = d$, and $\lambda(\rho) = (4\pi)^{d/2}\, e^{-\rho^2}$. That conditions (i)–(v) hold is fairly obvious. Condition (vi) holds with $\theta := 2$ and $a := 1$. Since $\lambda^{(k)} \in \lambda \Pi_k$, it is easy to see that condition (vii) is satisfied with $N := d + 1$. The desired conclusion now follows from Theorem 3.1 (with $\gamma_1 := \gamma$).

EXAMPLE 3.4. *Generalized Multiquadric.* Let $\gamma_0 > 0$ and define $\phi := (1 + |\cdot|^2)^{(\gamma_0 - d)/2}$ if $\gamma_0 - d \notin 2\mathbb{Z}_+$ or, $\phi := (1 + |\cdot|^2)^{(\gamma_0 - d)/2} \log(1 + |\cdot|^2)$ if $\gamma_0 + d \in 2\mathbb{Z}_+$. We will show, as an application of Theorem 3.1, that the stationary ladder $(S^h(\phi; \Xi))_h$ provides L_p-approximation of order γ_0 for all $1 \leqslant p \leqslant \infty$ whenever Ξ is a sufficiently small perturbation of $\mathbb{Z}^d$. Moreover, if κ: $(0, 1] \to (0, \infty)$ satisfies

$$\limsup_{h \to 0} \kappa(h) \log(1/h) < \frac{\pi}{\gamma_1}, \qquad \text{for some} \quad \gamma_1 \in (0, \infty),$$

and if $\phi_h := \phi(\kappa(h)\,\cdot\,)$, $\forall h \in (0, 1]$, then the non-stationary ladder $(S^h(\phi_h; \Xi))_h$ provides L_p-approximation of order $\gamma_0+\gamma_1$ for all $1 \leqslant p \leqslant \infty$ whenever Ξ is a sufficiently small perturbation of $\mathbb{Z}^d$.

For this we note that according to [16], $\hat{\phi}$ can be identified on $\mathbb{R}^d\backslash 0$ with $b|\cdot|^{-\gamma_0/2}\,K_{\gamma_0/2}(|\cdot|)$, where K_ν is the modified Bessel function of order ν (see [1]) and $b = b(d, \gamma_0)$ is some nonzero constant. One obtains from [1] that for $\nu > 0$,

$$K_\nu(\rho) = \rho^{-\nu} A_1(\rho^2) + \rho^\nu A_2(\rho^2) + \rho^\nu \log(\rho)\, A_3(\rho^2), \qquad \rho > 0,$$

where A_1, A_2, A_3 are entire and $A_1(0) \neq 0$. Actually, $A_3 \neq 0$ only when $\nu \in \mathbb{N}$. So, in terms of Theorem 3.1,

$$b^{-1}\lambda(\rho) = \rho^{\gamma_0/2} K_{\gamma_0/2}(\rho) = A_1(\rho^2) + \rho^{\gamma_0} A_2(\rho^2) + \rho^{\gamma_0} \log(\rho)\, A_3(\rho^2), \qquad \rho \geqslant 0. \tag{3.5}$$

Note that $\lambda(0) \neq 0$, $\lambda \in C([0, \infty)) \cap C^\infty((0, \infty))$, and $\bar{\mu} = \min\{\gamma_0, d\}$. Hence (i), (ii), and (iii) of Theorem 3.1 hold. If $0 < \varepsilon < \min\{1, \gamma_0\}$, then (iv) follows easily from (3.5). We turn now to conditions (v)–(vii). For this we employ the following integral representation of K_ν (see [1]). If $\nu > 0$, then

$$K_\nu(\rho) = \mathrm{const}(\nu)\, \rho^\nu \int_1^\infty e^{-\rho t}(t^2-1)^{\nu - 1/2}\, dt, \qquad \rho > 0.$$

Hence,

$$\lambda(\rho) = \pm\mathrm{const}(d, \gamma_0)\, \rho^{\gamma_0} \int_1^\infty e^{-\rho t}(t^2-1)^{(\gamma_0-1)/2}\, dt, \qquad \rho > 0. \tag{3.6}$$

Note that $|\lambda(\rho)| > 0$ for all $\rho \in [0, \infty)$. Put $\theta := 1$. Now if $a > 1$, then

$$\begin{aligned}\frac{|\lambda(\rho)|}{\exp(-a\rho)} &= \mathrm{const}(d, \gamma_0)\, \rho^{\gamma_0} \int_1^\infty e^{-\rho(t-a)}(t^2-1)^{(\gamma_0-1)/2}\, dt \\ &\geqslant \mathrm{const}(d, \gamma_0)\, \rho^{\gamma_0} \int_1^a e^{\rho(a-t)}(t^2-1)^{(\gamma_0-1)/2}\, dt \nearrow \infty \qquad \text{as} \quad \rho \nearrow \infty\end{aligned}$$

which proves (vi). Now, due to the exponential decay of the integrand in (3.6) when $\rho > 0$, it is a straightforward matter to verify that

$$\frac{d^k}{d\rho^k} \int_1^\infty e^{-\rho t}(t^2-1)^{(\gamma_0-1)/2}\, dt = \int_1^\infty \frac{d^k}{d\rho^k} e^{-\rho t}(t^2-1)^{(\gamma_0-1)/2}\, dt, \qquad k \in \mathbb{Z}_+.$$

Hence,

$$\frac{\lambda^{(k)}(\rho)}{\operatorname{const}(d,\gamma_0)} = \pm \sum_{j=0}^{k} \binom{k}{j} \gamma_0(\gamma_0-1)\cdots(\gamma_0-(k-j-1))\, \rho^{\gamma_0-(k-j)}$$
$$\times \int_1^\infty (-t)^j\, e^{-\rho t}(t^2-1)^{(\gamma_0-1)/2}\, dt.$$

Thus, for $\rho > 1$,

$$\begin{aligned} |\lambda^{(k)}(\rho)| &\leqslant \operatorname{const}(d,\gamma_0,k)\, \rho^{\gamma_0} \int_1^\infty t^k e^{-\rho t}(t^2-1)^{(\gamma_0-1)/2}\, dt \\ &\leqslant \operatorname{const}(d,\gamma_0,k)\, \rho^{\gamma_0} e^{-\rho} \int_1^\infty t^k e^{1-t}(t^2-1)^{(\gamma_0-1)/2}\, dt \\ &= \operatorname{const}(d,\gamma_0,k)\, \rho^{\gamma_0} e^{-\rho}. \end{aligned}$$

Therefore (vii) and (v) hold. The desired conclusion now follows from Theorem 3.1.

Another scenario where Theorem 5.8 can be applied is described in the following result.

THEOREM 3.7. *Let $\phi \in C(\mathbb{R}^d)$ be a radially symmetric function satisfying $|\cdot|^{d+1}\phi \in L_1$. Define $\lambda \in C^{d+1}[0,\infty)$ by $\hat{\phi}(x) = \lambda(|x|)$, $x \in \mathbb{R}^d$, and assume that for some $\gamma > d$,*

(i) $\sup_{0 \leqslant \rho < \infty}((1+\rho)^{-\gamma}/|\lambda(\rho)|) < \infty$ *and*
(ii) $|\lambda^{(k)}(\rho)| = O(\rho^{-\gamma-k})$ *as* $\rho \to \infty$, $\forall 0 \leqslant k \leqslant d+1$.

Let $\theta \in (0,1]$ and for $h \in (0,1]$ define $\phi_h := \phi(h^\theta \cdot)$. If Ξ is a sufficiently small perturbation of $\mathbb{Z}^d$, then $(S^h(\phi_h;\Xi))_h$ provides L_p-approximation of order $\theta\gamma$ for all $1 \leqslant p \leqslant \infty$.

Theorem 3.7 applies, for example, to the exponentially decaying function $\phi = |\cdot|^{(\gamma-d)/2} K_{(\gamma-d)/2}(|\cdot|)$ whose Fourier transform is a constant times $(1+|\cdot|^2)^{-\gamma/2}$. Furthermore, if we multiply this function by a radially symmetric $\sigma \in \mathscr{D}\backslash 0$, then Theorem 3.7 applies to the resultant compactly supported function $\phi = \sigma|\cdot|^{(\gamma-d)/2} K_{(\gamma-d)/2}(|\cdot|)$ provided σ has a non-negative Fourier transform. Regarding the applicability of Theorem 3.7 to Wendland's compactly supported radial functions $\phi_{d,k}$, it is easy to derive from [25] that for d odd, if γ is chosen to satisfy condition (i), then condition (ii) necessarily fails. One expects the same in the case d even, but this has yet to be proven.

4. SOME USEFUL LEMMATA

In this section we gather some technical lemmata which will be used in the following section. The following lemma shows that a weighted ℓ_p-norm is dominated by its corresponding weighted L_p-norm for band-limited functions (with a fixed band).

LEMMA 4.1. *Let $\rho: \mathbb{R}^d \to [1, \infty)$ be measurable, have at most polynomial growth at ∞, and satisfy*

$$\rho(x+y) \leqslant \rho(x)\,\rho(y), \qquad \forall x, y \in \mathbb{R}^d.$$

Then, for all $1 \leqslant p \leqslant \infty$,

$$\|\rho f\|_{\ell_p(\mathbb{Z}^d)} \leqslant \operatorname{const}(d, \rho)\,\|\rho f\|_{L_p(\mathbb{R}^d)},$$

whenever $f \in L_p$ and $\operatorname{supp} \hat{f} \subseteq 2\pi\bar{Q}$.

Proof. See [15, Lemma 1].

The following variant of Poisson's summation formula shows how the semi-discrete convolution acts in the Fourier transform domain.

LEMMA 4.2. *Let $\phi \in \hat{\mathscr{D}}$, and let f be a tempered distribution such that* $\operatorname{supp} \hat{f}$ *is compact. Then for all $h > 0$,*

$$(\phi *'_h f)^\wedge = \hat{\phi}(h\,\cdot) \sum_{j \in \mathbb{Z}^d} \hat{f}(\cdot - 2\pi j/h).$$

Proof. See [19, Lemma 5.7].

The following result allows us to work with a non-harmonic Fourier series in a way similar to that of the standard Fourier series provided that the frequencies in our nonharmonic Fourier series are a sufficiently small perturbation of $\mathbb{Z}^d$. We state the result in slightly more generality than needed only to suggest a useful formulation of the problem. The context in which we will actually use the lemma is mentioned in the forthcoming remark. We mention that a similar result can be derived from the results of [15].

LEMMA 4.3. *Let $\zeta \in \hat{\mathscr{D}}$ satisfy $\sum_{j \in \mathbb{Z}^d} \hat{\zeta}(\cdot + 2\pi j) = 1$ (or equivalently, $\zeta(j) = \delta_{0,j}$, $j \in \mathbb{Z}^d$). For $\xi \in \mathbb{R}^d$, let, $\hat{\zeta}_\xi$ be the $2\pi\mathbb{Z}^d$-periodic function defined by*

$$\hat{\zeta}_\xi(x) := \sum_{j \in \mathbb{Z}^d} e_\xi(x + 2\pi j)\,\hat{\zeta}(x + 2\pi j), \qquad x \in \mathbb{R}^d.$$

Let $\rho\colon \mathbb{Z}^d \to [1, \infty)$ *have at most polynomial growth and satisfy*

$$\rho(j+k) \leqslant \rho(j)\,\rho(k), \qquad \forall j, k \in \mathbb{Z}^d.$$

Then there exists $\delta(\zeta, \rho) > 0$ *such that if* $\xi_j \in j + \delta\bar{Q}$, $\forall j \in \mathbb{Z}^d$, *for some* $0 < \delta < \delta(\zeta, \rho)$, *then there exists a linear mapping* $\Lambda\colon \ell_\infty \to \ell_\infty$, *depending only on* ζ *and* $(\xi_j)_{j\in\mathbb{Z}^d}$, *such that*

(1) $\|\Lambda a\|_{\ell_1} \leqslant \mathrm{const}(d, \zeta, \delta)\,\|a\|_{\ell_1}$, $\forall a \in \ell_1$;

(2) $\sum_{j\in\mathbb{Z}^d} (\Lambda a)(j)\,\hat{\zeta}_{-\xi_j}(x) = \sum_{j\in\mathbb{Z}^d} a(j)\,e_{-j}(x)$, $\forall x \in \mathbb{R}^d$, $a \in \ell_1$.

Moreover, if $\omega\colon \mathbb{Z}^d \to [1, \infty)$ *satisfies*

(i) $\omega(j) \leqslant \rho(j)$, $\forall j \in \mathbb{Z}^d$;

(ii) $\omega(j+k) \leqslant \omega(j)\,\omega(k)$, $\forall j, k \in \mathbb{Z}^d$,

then for all $1 \leqslant p \leqslant \infty$,

(3) $\|\omega\Lambda a\|_{\ell_p} \leqslant \mathrm{const}(d, \zeta, \omega, \delta)\,\|\omega a\|_{\ell_p}$, $\forall a \in \ell_\infty$.

Remark 4.4. If $\operatorname{supp}\hat{\zeta} \subset [-\pi - \varepsilon_1, \pi + \varepsilon_1]^d$ and $\hat{\zeta} = 1$ on $[-\pi + \varepsilon_1, \pi - \varepsilon_1]^d$ for some $\varepsilon_1 \in (0, \pi)$, then $\hat{\zeta}_\xi = e_\xi$ on $[-\pi + \varepsilon_1, \pi - \varepsilon_1]^d$ for all $\xi \in \mathbb{R}^d$. Hence it follows from (2) that

$$\sum_{j\in\mathbb{Z}^d} (\Lambda a)(j)\,e_{-\xi_j}(x) = \sum_{j\in\mathbb{Z}^d} a(j)\,e_{-j}(x), \qquad \forall x \in [-\pi + \varepsilon_1, \pi - \varepsilon_1]^d, \quad a \in \ell_1. \tag{4.5}$$

In proving Lemma 4.3, we make essential use of the following well known result.

LEMMA 4.6. *Let* X *be a Banach space and let* $L\colon X \to X$ *be a bounded linear operator. If* $\|1 - L\| < 1$, *then* L *is boundedly invertible and*

$$\|L^{-1}\| \leqslant \frac{1}{1 - \|1 - L\|},$$

where $\|\ \|$ *denotes the operator norm in* X.

Proof of Lemma 4.3. For $\delta > 0$, define

$$N(\delta) := \sum_{j\in\mathbb{Z}^d} \rho(j)\,\|\delta_{j,0} - \zeta(\cdot + j)\|_{L_\infty(\delta Q)}.$$

Since ρ has at most polynomial growth, since ζ decays rapidly (being a member of $\hat{\mathscr{D}}$), and since each term in the sum defining $N(\delta)$ decreases to 0 as $\delta \to 0$, it follows by the Lebesgue Dominated Convergence Theorem that $N(\delta) \to 0$ as $\delta \to 0$. Hence, there exists $\delta(\zeta, \rho) > 0$ such that $N(\delta) < 1$ whenever $0 < \delta < \delta(\zeta, \rho)$. Let $\xi_j \in j + \delta \bar{Q}$, $j \in \mathbb{Z}^d$ for some $0 < \delta < \delta(\zeta, \rho)$. Define the linear operator $L: \ell_\infty \to \ell_\infty$ by

$$La(j) := \sum_{k \in \mathbb{Z}^d} a(k)\, \zeta(j - \xi_k), \qquad j \in \mathbb{Z}^d.$$

Let $\omega: \mathbb{Z}^d \to [1, \infty)$ satisfy (i) and (ii). For $1 \leqslant p \leqslant \infty$, let X_p be the Banach space consisting of all sequences $a: \mathbb{Z}^d \to \mathbb{C}$ for which $\|a\|_{X_p} := \|\omega a\|_{\ell_p} < \infty$.

Claim. *For $1 \leqslant p \leqslant \infty$, L is a boundedly invertible operator on X_p and*

$$\|L^{-1} a\|_{X_p} \leqslant \mathrm{const}(d, \zeta, \omega, \delta)\, \|a\|_{X_p}, \qquad \forall a \in X_p.$$

Proof. In view of Lemma 4.6, and since $N(\delta) < 1$, it suffices to show that

$$\|a - La\|_{X_p} \leqslant N(\delta)\, \|a\|_{X_p}, \qquad \forall a \in X_p. \tag{4.7}$$

If $a \in X_1$, then

$$\begin{aligned}
\|a - La\|_{X_1} &\leqslant \sum_{j \in \mathbb{Z}^d} \omega(j) \sum_{k \in \mathbb{Z}^d} |a(k)|\, |\delta_{k,j} - \zeta(j - \xi_k)| \\
&= \sum_{k \in \mathbb{Z}^d} \omega(k)\, |a(k)| \sum_{j \in \mathbb{Z}^d} \frac{\omega(j)}{\omega(k)}\, |\delta_{k,j} - \zeta(j - \xi_k)|, \\
&\qquad \text{by Fubini's Theorem,} \\
&= \sum_{k \in \mathbb{Z}^d} \omega(k)\, |a(k)| \sum_{j \in \mathbb{Z}^d} \frac{\omega(j+k)}{\omega(k)}\, |\delta_{j,0} - \zeta(j + k - \xi_k)| \\
&\leqslant \sum_{k \in \mathbb{Z}^d} \omega(k)\, |a(k)| \sum_{j \in \mathbb{Z}^d} \omega(j)\, \|\delta_{j,0} - \zeta(\cdot + j)\|_{L_\infty(\delta Q)} \\
&\leqslant N(\delta)\, \|a\|_{X_1} \qquad \text{by (i).}
\end{aligned}$$

If $a \in X_\infty$, then

$$\begin{aligned}
\|a - La\|_{X_\infty} &\leqslant \sup_{j \in \mathbb{Z}^d} \omega(j) \sum_{k \in \mathbb{Z}^d} |a(k)|\, |\delta_{k,j} - \zeta(j - \xi_k)| \\
&\leqslant \|a\|_{X_\infty} \sup_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \frac{\omega(j)}{\omega(k)} |\delta_{k,j} - \zeta(j - \xi_k)| \\
&\leqslant \|a\|_{X_\infty} \sup_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \frac{\omega(j)}{\omega(k+j)} \|\delta_{k,0} - \zeta(\cdot - k)\|_{L_\infty(\delta Q)} \\
&\leqslant \|a\|_{X_\infty} \sum_{k \in \mathbb{Z}^d} \omega(-k)\, \|\delta_{k,0} - \zeta(\cdot - k)\|_{L_\infty(\delta Q)} \leqslant N(\delta)\, \|a\|_{X_\infty}.
\end{aligned}$$

Having established (4.7) for $p = 1$ and $p = \infty$, we then obtain (4.7) for all $1 \leqslant p \leqslant \infty$ by interpolation (see [3, Theorem 3.6]).

With the Claim in view for the special case $\omega = 1$ and $p = \infty$, we define

$$\Lambda a := L^{-1} a, \qquad a \in \ell_\infty.$$

Note that Λ is a linear mapping of ℓ_∞ onto ℓ_∞, and since the definition of L depends only on ζ and $(\xi_j)_{j \in \mathbb{Z}^d}$, the same is true of Λ. Note that (3) follows from the Claim. Note that (1) follows from (3) in the special case $\omega = 1$ and $p = 1$. We turn now to (2). Let $a \in \ell_1$. By (1), $\Lambda a \in \ell_1$. Define

$$\psi := \sum_{j \in \mathbb{Z}^d} (\Lambda a)(j)\, \zeta(\cdot - \xi_j).$$

Then since $\Lambda a \in \ell_1$ and $\zeta \in L_1$, it follows that $\psi \in L_1$ and

$$\hat{\psi} = \hat{\zeta} \sum_{j \in \mathbb{Z}^d} (\Lambda a)(j)\, e_{-\xi_j}.$$

Similarly, since $a \in \ell_1$, it follows that $\zeta *' a \in L_1$ and

$$(\zeta *' a)^\wedge = \hat{\zeta} \sum_{j \in \mathbb{Z}^d} a(j)\, e_{-j}.$$

Note that for $j \in \mathbb{Z}^d$,

$$\psi(j) = \sum_{k \in \mathbb{Z}^d} (\Lambda a)(k)\, \zeta(j - \xi_k) = (L\Lambda a)(j) = a(j).$$

Therefore

$$\hat{\zeta} \sum_{j \in \mathbb{Z}^d} a(j)\, e_{-j} = (\zeta *' a)^\wedge = (\zeta *' \psi)^\wedge$$

$$= \hat{\zeta} \sum_{k \in \mathbb{Z}^d} \hat{\psi}(\cdot + 2\pi k), \qquad \text{by Lemma 4.2,}$$

$$= \hat{\zeta} \sum_{k \in \mathbb{Z}^d} \hat{\zeta}(\cdot + 2\pi k) \sum_{j \in \mathbb{Z}^d} (\Lambda a)(j)\, e_{-\xi_j}(\cdot + 2\pi k)$$

$$= \hat{\zeta} \sum_{j \in \mathbb{Z}^d} (\Lambda a)(j)\, \hat{\zeta}_{-\xi_j},$$

since $\Lambda a \in \ell_1$. Finally, we obtain (2) from the requirement $\sum_{j \in \mathbb{Z}^d} \hat{\zeta}(\cdot + 2\pi j) = 1$. ∎

When dealing with basis functions ϕ which have growth at ∞, a difficulty which invariably arises is that of identifying functions in $S(\phi; \Xi)$ by specifying their Fourier transform. The following lemma gives, under certain assumptions on ϕ, a simple solution to this difficulty. We mention that the set $(0, \gamma_0] \cup \{\gamma_0\}$, appearing below, equals $(0, \gamma_0]$ when $\gamma_0 > 0$ and equals $\{0\}$ when $\gamma_0 = 0$.

LEMMA 4.8. *Let $\phi \in C(\mathbb{R}^d)$ have at most polynomial growth at ∞. Assume that $\hat{\phi}$ can be identified on $\mathbb{R}^d \backslash 0$ with $|\cdot|^{-\gamma_0} \lambda$, where $\gamma_0 \geqslant 0$ and λ: $\mathbb{R}^d \to \mathbb{C}$ is locally integrable on $\mathbb{R}^d$, continuous on a neighborhood of 0, and satisfies $\lambda(0) \neq 0$. Assume that there exists $\mu \in (0, \gamma_0] \cup \{\gamma_0\}$ such that*

$$|\phi(x)| = o(|x|^{\gamma_0 - \mu}) \qquad \text{as} \quad |x| \to \infty.$$

Let $\Xi \subset \mathbb{R}^d$, $b \in \ell_1(\Xi)$, and define

$$\hat{g}(x) := |x|^{-\gamma_0}\, \lambda(x) \sum_{\xi \in \Xi} b(\xi)\, e_{-\xi}(x), \qquad x \in \mathbb{R}^d \backslash 0.$$

If $\hat{g}$ can be identified a.e. as the Fourier transform of a function $g \in L_1$, and if

$$\sum_{\xi \in \Xi} (1 + |\xi|)^{\gamma_0 - \mu}\, |b(\xi)| < \infty, \tag{4.9}$$

then $g = \sum_{\xi \in \Xi} b(\xi)\, \phi(\cdot - \xi)$.

We remark that under much weaker assumptions than $g \in L_1$, there is a standard argument which concludes that g and $\sum_{\xi \in \Xi} b(\xi)\, \phi(\cdot - \xi)$ differ by at most a polynomial. The strong assumption $g \in L_1$ (which will suffice us in the sequel) serves as a simple means of ensuring that the errant polynomial is in fact 0.

Proof. By (4.9) and since $|\phi(x)| = O(|x|^{\gamma_0 - \mu})$ it follows that the sum

$$f := \sum_{\xi \in \Xi} b(\xi)\, \phi(\cdot - \xi)$$

converges in the space of tempered distributions. We begin by showing that $\hat{g} = \hat{f}$ on $\mathbb{R}^d \backslash 0$. For that let $\psi \in \mathscr{D}$ be such that $\operatorname{supp} \psi \subset \mathbb{R}^d \backslash 0$. Then

$$\begin{aligned}
\langle \psi, \hat{g} \rangle &= \int_{\operatorname{supp} \psi} \psi(x)\, |x|^{-\gamma_0}\, \lambda(x) \sum_{\xi \in \Xi} b(\xi)\, e_{-\xi}(x)\, dx \\
&= \sum_{\xi \in \Xi} b(\xi) \int_{\operatorname{supp} \psi} \psi(x)\, |x|^{-\gamma_0}\, \lambda(x)\, e_{-\xi}(x)\, dx, \qquad \text{since} \quad b \in \ell_1(\Xi), \\
&= \sum_{\xi \in \Xi} b(\xi) \langle \hat{\psi}, \phi(\cdot - \xi) \rangle = \langle \hat{\psi}, f \rangle = \langle \psi, \hat{f} \rangle.
\end{aligned}$$

Therefore $\hat{g} = \hat{f}$ on $\mathbb{R}^d \backslash 0$, and hence $f - g$ is a polynomial. If $\gamma_0 = 0$, then $\gamma_0 - \mu = 0$ and so by (4.9), $|f(x)| = o(1)$ as $|x| \to \infty$; since $g \in L_1$, we must have $f = g$. Having dispensed with the case $\gamma_0 = 0$, let us assume that $\gamma_0 > 0$ (which implies $\mu > 0$). Since $g \in L_1$, in order to show that $\hat{f} = \hat{g}$ (and hence prove the lemma), it suffices to show that $\hat{f}$ is regular (i.e., locally integrable) on some neighborhood of the origin. We will accomplish this by showing that there exists an $\varepsilon_1 > 0$, $F \in L_1(\varepsilon_1 B/2)$, and a sequence $(f_n)_{n \in \mathbb{N}}$ in L_1 such that $\hat{f}_n \to \hat{f}$ in the space of tempered distributions, and $|\hat{f}_n(x)| \leqslant cF(x)$ for all $x \in (\varepsilon_1/2)\, B \backslash 0$, $n \in \mathbb{N}$, for some $c < \infty$ which does not depend on n or x.

There exists $\varepsilon_1, c_1, c_2 \in (0, \infty)$ such that $c_1 \leqslant |\lambda(x)| \leqslant c_2$ $\forall x \in \varepsilon_1 B$. Define $F := 1 + |\cdot|^{-d+\mu}$. Note that $F \in L_1(\varepsilon_1 B/2)$. Let $v \in \hat{\mathscr{D}}$ be such that $v(0) = 1$, $\hat{v} \geqslant 0$, and $\operatorname{supp} \hat{v} \subset \varepsilon_1 B/2$. For $n \in \mathbb{N}$, define

$$f_n := \sum_{\xi \in \Xi} b(\xi)\, v((\cdot - \xi)/n)\, \phi(\cdot - \xi).$$

By (4.9), and since $v(0) = 1$, it follows that $f_n \to f$ in the space of tempered distributions. Therefore, $\hat{f}_n \to \hat{f}$ in the space of tempered distributions. On the other hand, since $b \in \ell_1(\Xi)$ and $v(\cdot/n)\, \phi \in L_1$, it follows that $f_n \in L_1$ and for $x \in \varepsilon_1 B \backslash 0$,

$$\hat{f}_n(x) = (v(\cdot/n)\, \phi)^\wedge\, (x) \sum_{\xi \in \Xi} b(\xi)\, e_{-\xi}(x).$$

Note that for $x \in \varepsilon_1 B \backslash 0$, $|\sum_{\xi \in \Xi} b(\xi)\, e_{-\xi}(x)| \leqslant (\|g\|_{L_1}/c_1)\, |x|^{\gamma_0}$. Therefore,

$$|\hat{f}_n(x)| \leqslant \frac{\|g\|_{L_1}}{c_1}\, |(v(\cdot/n)\, \phi)^\wedge\, (x)|\, |x|^{\gamma_0}, \qquad \forall x \in \varepsilon_1 B \backslash 0.$$

So, in order to establish $|f_n(x)| \leqslant cF(x)\ \forall x \in (\varepsilon/2)\, B\backslash 0$, and hence prove the lemma, it suffices to show that

$$|(v(\cdot/n)\,\phi)\hat{}\ (x)| \leqslant c(|x|^{-\gamma_0} + |x|^{-d-\gamma_0+\mu}) \qquad \text{for all} \quad x \in \frac{\varepsilon}{2}\, B\backslash 0. \tag{4.10}$$

Since $v \in \hat{\mathscr{D}}$ and ϕ satisfies $|\phi(x)| = O(|x|^{\gamma_0-\mu})$ as $|x| \to \infty$, it follows that $\|v(\cdot/n)\,\phi\|_{L_1} = O(n^{d+\gamma_0-\mu})$ as $n \to \infty$. Using the estimate $|(v(\cdot/n)\,\phi)\hat{}\ (x)| \leqslant \|v(\cdot/n)\,\phi\|_{L_1}$, we thus obtain (4.10) for the case $0 < |x| \leqslant \varepsilon_1/n$. For the remaining case, $\varepsilon_1/n < |x| \leqslant \varepsilon_1/2$ we have

$$\begin{aligned}|(v(\cdot/n)\,\phi)\hat{}\ (x)| &= (2\pi)^{-d}\, |(n^d\hat{v}(n\,\cdot) * \hat{\phi})(x)| \leqslant \|\,|\cdot|^{-\gamma_0}\,\lambda\|_{L_\infty(x+(\varepsilon_1/2n)\,B)}\\ &\leqslant c_2\left(|x| - \frac{\varepsilon_1}{2n}\right)^{-\gamma_0} \leqslant c_2 2^{\gamma_0}\, |x|^{-\gamma_0}. \qquad \blacksquare\end{aligned}$$

5. THE GENERAL RESULTS

The foundation of our approach might well be called *approximation by replacement*. Since the structure of $S^h(\phi_h; \Xi)$ is irrelevant to this technique, we will, for the moment, simply assume that $(S^h)_{h\in(0\ldots h_0]}$ is a family of closed subspaces of $C(\mathbb{R}^d)$ (these will eventually correspond to $S(\phi_h; \Xi)$), and we define as usual

$$S_h^h := \{s(\cdot/h)\colon s \in S_h\}, \qquad h \in (0, h_0].$$

Beginning with the observation that if $h = 2^{-n}$, and $f \in B_p^{\gamma, 1}$, then

$$f \approx \sum_{k=0}^{n} \sum_{j\in\mathbb{Z}^d} f_k(2^{-k}j)\, \eta(2^k\cdot - j)$$

is a good approximation of f, the idea is to replace each $\eta(2^k\cdot - j)$ with an approximation drawn from S_h^h. In other words, we seek suitable $q_{k,j} \in S_h^h$ such that

$$f \approx \sum_{k=0}^{n} \sum_{j\in\mathbb{Z}^d} f_k(2^{-k}j)\, q_{k,j}$$

is also a good approximation to f. In order to carry the error analysis through, the issue becomes not so much how well each $\eta(2^k\cdot - j)$ is approximated by $q_{k,j}$, but rather how well, for each k, the mapping

$$\ell_p \ni c \mapsto \sum_{j\in\mathbb{Z}^d} c(j)\, \eta(2^k\cdot - j) \in L_p$$

is approximated by the mapping

$$\ell_p \ni c \mapsto \sum_{j \in \mathbb{Z}^d} c(j)\, q_{k,j} \in L_p.$$

The following definition and lemma provide a simple means for measuring the size of (or closeness of) such mappings.

DEFINITION 5.1. We define $\mathcal{N}$ to be the collection of all sequences $(\mathbf{f}_j)_{j \in \mathbb{Z}^d}$ in $C(\mathbb{R}^d)$ for which

$$\sum_{j \in \mathbb{Z}^d} \|\mathbf{f}_j\|_{L_\infty(K)} < \infty \qquad \text{for all compact} \quad K \subset \mathbb{R}^d,$$

and

$$\|\mathbf{f}\|_{\mathcal{N}} := \max \left\{ \sup_{j \in \mathbb{Z}^d} \|\mathbf{f}_j\|_{L_1}, \left\| \sum_{j \in \mathbb{Z}^d} |\mathbf{f}_j| \right\|_{L_\infty} \right\} < \infty.$$

For any complex valued function g whose domain contains $\mathbb{Z}^d$, we define formally

$$\mathbf{f} \cdot g := \sum_{j \in \mathbb{Z}^d} g(j)\, \mathbf{f}_j.$$

LEMMA 5.2. *Let $\mathbf{f} \in \mathcal{N}$. If $c \in \ell_\infty$, then the sum $\mathbf{f} \cdot c$ converges unconditionally in $C(\mathbb{R}^d)$. Moreover, for all $1 \leqslant p \leqslant \infty$, the mapping $c \mapsto \mathbf{f} \cdot c$ is a bounded linear mapping from ℓ_p into L_p and as such its norm does not exceed $\|\mathbf{f}\|_{\mathcal{N}}$.*

Proof. That the sum $\mathbf{f} \cdot c$ converges unconditionally in $C(\mathbb{R}^d)$ whenever $c \in \ell_\infty$ is an immediate consequence of the requirement that $\sum_{j \in \mathbb{Z}^d} \|\mathbf{f}_j\|_{L_\infty(K)} < \infty$ for all compact $K \in \mathbb{R}^d$. That the lemma is true for $p = 1$ and $p = \infty$ is clear from the definition of the $\mathcal{N}$-norm. We then interpolate to obtain the lemma for all $1 \leqslant p \leqslant \infty$ (see [3, Theorem 3.6]). ▮

We now state the theorem which provides the foundation of our approach.

THEOREM 5.3. *Let $(S_r)_{r \in (0, h_0]}$ be a family of closed subspaces of $C(\mathbb{R}^d)$, and define*

$$S_r^h := \{ s(\cdot/h) \colon s \in S_r \}, \qquad \forall h, r \in (0, h_0].$$

Let $\eta \in \hat{\mathscr{D}}$ *and* $\varepsilon \in (0, 2\pi)$ *be such that* $\operatorname{supp} \hat{\eta} \subset \varepsilon Q$ *and* $\hat{\eta} = 1$ *on* $\frac{1}{2}\varepsilon Q$. *Put* $\boldsymbol{\eta}_j := \eta(\cdot - j)$, $j \in \mathbb{Z}^d$. *If there exists* $\gamma > 0$ *such that for some* $A < \infty$,

$$\operatorname{dist}(\boldsymbol{\eta}, (S_r^h)^{\mathbb{Z}^d} \cap \mathscr{N}; \mathscr{N}) < A h^\gamma, \qquad \forall 0 < r \leqslant h \leqslant h_0, \tag{5.4}$$

then

$$\operatorname{dist}(f, S_h^h; L_p) \leqslant (1 + \operatorname{const}(d, \gamma)\, A)\, h^\gamma \, \|f\|_{B_p^{\gamma, 1}},$$

for all $f \in B_p^{\gamma, 1}$, $1 \leqslant p \leqslant \infty$.

Proof. Without loss of generality assume $h_0 = 1$. Let $\gamma > 0$ and assume that (5.4) holds. Let $1 \leqslant p \leqslant \infty$. Let $f \in B_p^{\gamma, 1}$, and let f_k be as in (2.1), $k \in \mathbb{Z}_+$. For $h \in (0, 1]$, let $n := n(h)$ be the largest integer for which $h 2^n \leqslant 1$. First, let us make three observations:

Claim 5.5. *For all* $h \in (0, 1]$,

(1) $f_k = \eta *'_{h2^{n-k}} f_k$, $\forall k \in \mathbb{Z}_+$;
(2) $(h 2^{n-k})^{d/p} \, \|f_k\|_{\ell_p(h 2^{n-k} \mathbb{Z}^d)} \leqslant \operatorname{const}(d) \, \|f_k\|_{L_p}$, $\forall k \in \mathbb{Z}_+$;
(3) $\|f - \sum_{k=0}^n f_k\|_{L_p} \leqslant \|f\|_{B_p^{\gamma, 1}} h^\gamma$.

Proof. Note that $\operatorname{supp} \hat{f}_k$ is compact. Hence, by Lemma 4.2,

$$(\eta *'_{h2^{n-k}} f_k)^\wedge = \hat{\eta}(h 2^{n-k} \cdot) \sum_{j \in \mathbb{Z}^d} \hat{f}_k(\cdot - 2\pi j/(h 2^{n-k})).$$

By (2.1), $\operatorname{supp} \hat{f}_k \subseteq \operatorname{supp} \hat{\eta}(2^{1-k} \cdot) \subseteq 2^{k-1} \varepsilon Q$, $\forall k \in \mathbb{Z}_+$. It is now a straightforward matter to verify that $\hat{\eta}(h 2^{n-k} \cdot)$ and $\hat{f}_k(\cdot - 2\pi j/(h 2^{n-k}))$ have disjoint supports whenever $j \in \mathbb{Z}^d \backslash 0$ and that $\hat{\eta}(h 2^{n-k} \cdot) = 1$ on the support of $\hat{f}_k$. Therefore, $(\eta *'_{h2^{n-k}} f_k)^\wedge = \hat{f}_k$ which proves (1). Since $\operatorname{supp}(f_k(h 2^{n-k} \cdot))^\wedge \subseteq h 2^{n-k} 2^{k-1} \varepsilon Q \subset 2\pi Q$, it follows by Lemma 4.1 (with $\rho = 1$) that

$$\begin{aligned} \|f_k\|_{\ell_p(h 2^{n-k} \mathbb{Z}^d)} &= \|f_k(h 2^{n-k} \cdot)\|_{\ell_p(\mathbb{Z}^d)} \leqslant \operatorname{const}(d) \, \|f_k(h 2^{n-k} \cdot)\|_{L_p} \\ &= \operatorname{const}(d) (h 2^{n-k})^{-d/p} \, \|f_k\|_{L_p} \end{aligned}$$

which proves (2). Noting that $f = \sum_{k=0}^\infty f_k$, we obtain

$$\left\| f - \sum_{k=0}^n f_k \right\|_{L_p} \leqslant \sum_{k=n+1}^\infty \|f_k\|_{L_p} \leqslant 2^{-(n+1)\gamma} \sum_{k=n+1}^\infty 2^{k\gamma} \, \|f_k\|_{L_p} \leqslant h^\gamma \, \|f\|_{B_p^{\gamma, 1}}$$

which proves (3) and completes the proof of the claim.

It is convenient to define the scaling operator $\boldsymbol{\sigma}_h$ for $h > 0$ as

$$\begin{aligned} \boldsymbol{\sigma}_h f &:= f(\cdot / h), && \text{if } f\colon \mathbb{R}^d \to \mathbb{C}; \\ \boldsymbol{\sigma}_h \mathbf{f} &:= (\boldsymbol{\sigma}_h(\mathbf{f}_j))_{j \in \mathbb{Z}^d}, && \text{if } \mathbf{f} \in \mathscr{N}. \end{aligned}$$

By (5.4) there exists $\mathbf{g}^k = (\mathbf{g}_j^k)_{j\in\mathbb{Z}^d} \in (S_h)^{\mathbb{Z}^d} \cap \mathcal{N}$, $0 \leqslant k \leqslant n$, such that

$$\|\boldsymbol{\sigma}_{2^{k-n}}\mathbf{g}^k - \boldsymbol{\eta}\|_{\mathcal{N}} \leqslant A2^{\gamma(k-n)}, \qquad 0 \leqslant k \leqslant n. \tag{5.6}$$

(Note: The $2^{(k-n)}$ is playing the role of h in (5.4), while h is playing the role of r in (5.4). Inequality (5.6) is a valid application of (5.4) because $0 < h \leqslant 2^{(k-n)} \leqslant 1$.) Note that for $0 \leqslant k \leqslant n$, $\boldsymbol{\sigma}_h\mathbf{g}^k \in (S_h^h)^{\mathbb{Z}^d} \cap \mathcal{N}$ and it follows from Lemma 5.2 and from the assumption that S_h^h is a closed subspace of $C(\mathbb{R}^d)$ that $(\boldsymbol{\sigma}_h\mathbf{g}^k)\cdot c \in S_h^h$ for all $c \in \ell_\infty$. Therefore, by Claim 5.5 (2),

$$s_h := \sum_{k=0}^{n} (\boldsymbol{\sigma}_h\mathbf{g}^k)\cdot(\boldsymbol{\sigma}_{h^{-1}2^{k-n}} f_k) \in S_h^h.$$

Now,

$$\begin{aligned}
&\left\| s_h - \sum_{k=0}^{n} f_k \right\|_{L_p} \\
&\quad = \left\| \sum_{k=0}^{n} (\boldsymbol{\sigma}_h\mathbf{g}^k - \boldsymbol{\sigma}_{h2^{n-k}}\boldsymbol{\eta})\cdot(\boldsymbol{\sigma}_{h^{-1}2^{k-n}} f_k) \right\|_{L_p}, \qquad \text{by Claim 5.5 (1)},\\
&\quad \leqslant \sum_{k=0}^{n} (h2^{n-k})^{d/p}\, \|(\boldsymbol{\sigma}_{2^{k-n}}\mathbf{g}^k - \boldsymbol{\eta})\cdot(\boldsymbol{\sigma}_{h^{-1}2^{k-n}} f_k)\|_{L_p}\\
&\quad \leqslant \sum_{k=0}^{n} (h2^{n-k})^{d/p}\, \|\boldsymbol{\sigma}_{2^{k-n}}\mathbf{g}^k - \boldsymbol{\eta}\|_{\mathcal{N}}\, \|f_k\|_{\ell_p(h2^{n-k}\mathbb{Z}^d)}, \qquad \text{by Lemma 5.2},\\
&\quad \leqslant \sum_{k=0}^{n} A2^{\gamma(k-n)}\text{const}(d)\, \|f_k\|_{L_p}, \qquad \text{by (5.6) and Claim 5.5 (2)},\\
&\quad = \text{const}(d)\, A2^{-n\gamma} \sum_{k=0}^{n} 2^{k\gamma}\, \|f_k\|_{L_p} \leqslant \text{const}(d,\gamma)\, A\, \|f\|_{B_p^{\gamma,1}}\, h^\gamma.
\end{aligned}$$

Thus, with Claim 5.5 (3) in view, the theorem is proved. ∎

Returning to our original concern of approximation from $S^h(\phi_h; \Xi)$ we have the following which is an immediate consequence of Theorem 5.3 (with $S_r := S(\phi_r; \Xi)$).

COROLLARY 5.7. *Let $(\phi_h)_{h\in(0\ldots h_0]}$ be a family of functions in $C(\mathbb{R}^d)$. Let $\eta \in \hat{\mathscr{D}}$ and $\varepsilon \in (0, 2\pi)$ be such that* $\operatorname{supp}\hat{\eta} \subset \varepsilon Q$ *and* $\hat{\eta} = 1$ *on* $\frac{1}{2}\varepsilon Q$. *Put $\boldsymbol{\eta}_j := \eta(\cdot - j)$, $j \in \mathbb{Z}^d$. Let $\Xi \subset \mathbb{R}^d$. If there exists $\gamma > 0$ such that*

$$\sup_{0<r\leqslant h} \operatorname{dist}(\boldsymbol{\eta}, (S^h(\phi_r; \Xi))^{\mathbb{Z}^d} \cap \mathcal{N}; \mathcal{N}) = O(h^\gamma), \qquad \text{as} \quad h \to 0,$$

then $(S^h(\phi_h; \Xi))_h$ *provides* L_p*-approximation of order* γ *(in the sense of Definition* 2.2*) for all* $1 \leqslant p \leqslant \infty$

We now state the main result of this section. As mentioned before, the set $(0, \gamma_0] \cup \{\gamma_0\}$ equals $(0, \gamma_0]$ when $\gamma_0 > 0$ and equals $\{0\}$ when $\gamma_0 = 0$.

THEOREM 5.8. *Let* $(\phi_h)_{h \in (0, h_0]}$ *be a family of functions in* $C(\mathbb{R}^d)$ *with at most polynomial growth at* ∞, *and assume that there exists* $\gamma_0 \geqslant 0$ *such that for each* $h \in (0, h_0]$, *there exists a locally integrable function* λ_h *such that* $\hat{\phi}_h$ *can be identified on* $\mathbb{R}^d \backslash 0$ *with* $|\cdot|^{-\gamma_0} \lambda_h$. *Assume that there exists* $\varepsilon \in (0, 2\pi)$ *such that* $\lambda_h \in C(\varepsilon Q)$ *and* $|\lambda_h| > 0$ *on* εQ, $\forall h \in (0, h_0]$. *Let* $\eta \in \hat{\mathcal{D}}$ *be such that* $\operatorname{supp} \hat{\eta} \subset \varepsilon Q$ *and* $\hat{\eta} = 1$ *on* $\frac{1}{2} \varepsilon Q$. *Assume that there exists* $\mu \in (0, \gamma_0] \cup \{\gamma_0\}$ *such that for all* $0 < r \leqslant h \leqslant 1$,

(i) $|\phi_h(x)| = o(|x|^{\gamma_0 - \mu})$ *as* $|x| \to \infty$;

(ii) $(1 + |\cdot|)^{\gamma_0 - \mu} \, (\hat{\eta}(\,\cdot\,/h) \, |\cdot|^{\gamma_0} / \lambda_r)^\vee \in L_1$.

Let $\sigma \in \mathcal{D}$ *satisfy* $\operatorname{supp} \sigma \subset 2\pi Q$ *and* $\sigma = 1$ *on* εQ. *If there exists* $\gamma \in (0, \infty)$ *such that*

$$\sup_{0 < r \leqslant h} \left\| \left(\frac{\hat{\eta}(\,\cdot\,/h) \, |\cdot|^{\gamma_0}}{\lambda_r} \right)^\vee \right\|_{L_1} \sum_{j \in \mathbb{Z}^d} \| ((1 - \sigma) \, |\cdot|^{-\gamma_0} \, \lambda_r)^\vee \|_{L_\infty(j + Q)} = O(h^\gamma),$$
$$\text{as} \quad h \to 0,$$

then $(S^h(\phi_h; \Xi))_h$ *provides* L_p*-approximation of order* γ *(in the sense of Definition* 2.2*) for all* $1 \leqslant p \leqslant \infty$ *whenever* Ξ *is a sufficiently small perturbation of* $\mathbb{Z}^d$.

Conditions (i), (ii) serve to ensure that a certain approximant actually belongs to $S^h(\phi_h; \Xi)$. As far as the approximation order is concerned, the item of significance is the behavior of $\Gamma(r, h)$ as $r \leqslant h \to 0$, where

$$\Gamma(r, h) := \left\| \left(\frac{\hat{\eta}(\,\cdot\,/h) \, |\cdot|^{\gamma_0}}{\lambda_r} \right)^\vee \right\|_{L_1} \sum_{j \in \mathbb{Z}} \| ((1 - \sigma) \, |\cdot|^{-\gamma_0} \, \lambda_r)^\vee \|_{L_\infty} (j + Q).$$

Note that there are two factors in the definition of $\Gamma(r, h)$. In the stationary case, the second factor is fixed (independent of r and h) and so it is useful only when it is 0; the significance of the first factor,

$$\left\| \left(\frac{\hat{\eta}(\,\cdot\,/h) \, |\cdot|^{\gamma_0}}{\lambda} \right)^\vee \right\|_{L_1} = h^{\gamma_0} \left\| \left(\frac{\hat{\eta} \, |\cdot|^{\gamma_0}}{\lambda(h\,\cdot\,)} \right)^\vee \right\|_{L_1}$$

is that it is $O(h^{\gamma_0})$ if $(\hat{\eta}/\lambda(h\,\cdot\,))^\vee \in L_1$ for sufficiently small $h > 0$. In the non-stationary case, the second factor is usually most responsible for the decay of $\Gamma(r, h)$.

In view of Corollary 5.7, in order to prove Theorem 5.8, it suffices to prove the following:

LEMMA 5.9. *Under the hypothesis of Theorem* 5.8, *there exists* $\delta_0 > 0$ *such that if* $\boldsymbol{\delta}(\Xi) \leqslant \delta_0$, *then*

$$\operatorname{dist}(\boldsymbol{\eta}, (S^h(\phi_r; \Xi))^{\mathbb{Z}^d} \cap \mathcal{N}; \mathcal{N})$$
$$\leqslant \operatorname{const}(d, \delta_0) \left\| \left(\frac{\hat{\eta}(\,\cdot\,/h)\, |\cdot|^{\gamma_0}}{\lambda_r} \right)^{\vee} \right\|_{L_1} \sum_{j \in \mathbb{Z}^d} \|((1-\sigma)\, |\cdot|^{-\gamma_0}\, \lambda_r)^{\vee}\|_{L_\infty(j+Q)},$$

for all $0 < r \leqslant h \leqslant h_0$.

Proof. Put $\tau_{r,h} := (\hat{\eta}(\,\cdot\,/h)\, |\cdot|^{\gamma_0}/\lambda_r)^{\vee}$ and $\psi_r := ((1-\sigma)\, |\cdot|^{-\gamma_0}\, \lambda_r)^{\vee}$. Without loss of generality we may assume that $\sum_{j \in \mathbb{Z}^d} \|\psi_r\|_{L_\infty(j+Q)} < \infty$ and $h_0 = 1$. Define $\rho := (1 + |\cdot|)^{\gamma_0 - \mu}$, and note that $1 \leqslant \rho(j+k) \leqslant \rho(j)\, \rho(k)$ for all $j, k \in \mathbb{Z}^d$. There exists $\varepsilon_1 \in (0, \pi)$ such that $\operatorname{supp} \sigma \subset [-\pi + \varepsilon_1, \pi - \varepsilon_1]^d$. Let $\zeta \in \hat{\mathcal{D}}$ satisfy $\operatorname{supp} \hat{\zeta} \subset [-\pi - \varepsilon_1, \pi + \varepsilon_1]^d$, $\hat{\zeta} = 1$ on $[-\pi + \varepsilon_1, \pi - \varepsilon_1]^d$, and $\sum_{j \in \mathbb{Z}^d} \hat{\zeta}(\,\cdot\, + 2\pi j) = 1$. Let $\delta(\zeta, \rho)$ be as in Lemma 4.3, and let $\delta_0 \in (0, \delta(\zeta, \rho))$. Fix $0 < r \leqslant h \leqslant 1$, and let Ξ be any perturbation of $\mathbb{Z}^d$ satisfying $\boldsymbol{\delta}(\Xi) \leqslant \delta_0$. Using the countable axiom of choice,[1] there exists a sequence $(\xi_j)_{j \in \mathbb{Z}^d}$ with the property that $\xi_j \in (j + \delta_0 \bar{Q}) \cap \Xi$ for all $j \in \mathbb{Z}^d$. Let Λ be as in Lemma 4.3, and define

$$a_k(j) := h^{-d} \tau_{r,h}(j - k/h), \qquad j, k \in \mathbb{Z}^d;$$
$$b_k := \Lambda a_k, \qquad k \in \mathbb{Z}^d.$$

Note that by (ii) of Theorem 5.8 and Lemma 4.1, it follows that $\rho a_k \in \ell_1$ and hence b_k is well defined. By Lemma 4.3 (3),

$$\|\rho b_k\|_{\ell_1} \leqslant \operatorname{const}(d, \zeta, \rho, \delta_0)\, \|\rho a_k\|_{\ell_1}, \qquad \forall k \in \mathbb{Z}^d. \tag{5.10}$$

Hence by (i) of Theorem 5.8,

$$g_k := \sum_{j \in \mathbb{Z}^d} b_k(j)\, \phi_r(\,\cdot\,/h - \xi_j) \in S^h(\phi_r; \Xi), \qquad \forall k \in \mathbb{Z}^d.$$

CLAIM 5.11.

$$g_k = \sum_{j \in \mathbb{Z}^d} b_k(j)\, \psi_r(\,\cdot\,/h - \xi_j) + \eta(\,\cdot\, - k), \qquad \forall k \in \mathbb{Z}^d.$$

[1] If Ξ is locally finite, then it is not necessary to use the countable axiom of choice here, since for each $j \in \mathbb{Z}^d$, we could then define ξ_j to be the unique element of the finite set $\Xi \cap (j + \delta_0 \bar{Q})$ which is least in a lexicographical ordering of $\mathbb{R}^d$.

Proof. Fix $k \in \mathbb{Z}^d$ and put $g := \sum_{j \in \mathbb{Z}^d} b_k(j)\, \psi_r(\cdot - \xi_j) + \eta(h \cdot - k)$. Since $g \in L_1$ (as $b_k \in \ell_1$ and $\psi_r \in L_1$) and with Lemma 4.8 in view, in order to prove the claim, it suffices to show that

$$\hat{g}(x) = |x|^{-\gamma_0}\, \lambda_r(x) \sum_{j \in \mathbb{Z}^d} b_k(j)\, e_{-\xi_j}(x), \qquad \forall x \in \mathbb{R}^d \backslash 0. \tag{5.12}$$

First note that

$$\begin{aligned} \hat{g} &= h^{-d} e_{-k/h} \hat{\eta}(\cdot/h) + \hat{\psi}_r \sum_{j \in \mathbb{Z}^d} b_k(j)\, e_{-\xi_j} \\ &= h^{-d} e_{-k/h} \hat{\eta}(\cdot/h) + (1-\sigma)\, |\cdot|^{-\gamma_0}\, \lambda_r \sum_{j \in \mathbb{Z}^d} b_k(j)\, e_{-\xi_j}. \end{aligned}$$

Since $\sigma = 1$ on $\operatorname{supp} \hat{\eta}$ and $\sigma = 0$ outside of $[-\pi + \varepsilon_1, \pi - \varepsilon_1]^d$, in order to establish (5.12), and hence prove the claim, it suffices to show that

$$\sum_{j \in \mathbb{Z}^d} b_k(j)\, e_{-\xi_j}(x) = h^{-d} e_{-k/h}(x)\, \frac{\hat{\eta}(x/h)\, |x|^{\gamma_0}}{\lambda_r(x)}, \qquad \forall x \in [-\pi + \varepsilon_1, \pi - \varepsilon_1]^d.$$

For that let $x \in [-\pi + \varepsilon_1, \pi - \varepsilon_1]^d$. Note that on the one hand,

$$\begin{aligned} &(\zeta *' (h^{-d} \tau_{r,h}(\cdot - k/h))^\wedge (x) \\ &\quad = h^{-d} \hat{\zeta}(x) \sum_{j \in \mathbb{Z}^d} (\tau_{r,h}(\cdot - k/h))^\wedge (x - 2\pi j), \qquad \text{by Lemma 4.2,} \\ &\quad = h^{-d} \sum_{j \in \mathbb{Z}^d} e_{-k/h}(x + 2\pi j)\, \hat{\tau}_{r,h}(x + 2\pi j), \\ &\qquad \text{since} \quad \hat{\zeta} = 1 \text{ on } [-\pi + \varepsilon_1, \pi - \varepsilon_1]^d, \\ &\quad = h^{-d} e_{-k/h}(x)\, \frac{\hat{\eta}(x/h)\, |x|^{\gamma_0}}{\lambda_r(x)}, \qquad \text{since} \quad \operatorname{supp} \hat{\tau}_{r,h} \subset [-\pi + \varepsilon_1, \pi - \varepsilon_1]^d. \end{aligned}$$

While on the other hand,

$$\begin{aligned} (\zeta *' (h^{-d} \tau_{r,h}(\cdot - k/h)))^\wedge (x) &= \left(\sum_{j \in \mathbb{Z}^d} a_k(j)\, \zeta(\cdot - j) \right)^\wedge (x) \\ &= \hat{\zeta}(x) \sum_{j \in \mathbb{Z}^d} a_k(j)\, e_{-j}(x) \\ &= \sum_{j \in \mathbb{Z}^d} a_k(j)\, e_{-j}(x) \\ &= \sum_{j \in \mathbb{Z}^d} b_k(j)\, e_{-\xi_j}(x), \end{aligned}$$

by Remark 4.4 (as $x \in [-\pi + \varepsilon_1, \pi - \varepsilon_1]^d$). Hence the claim.

Define $\mathbf{g} := (g_k)_{k \in \mathbb{Z}^d} \in S^h(\phi_r; \Xi)^{\mathbb{Z}^d}$. Then by Claim 5.11,

$$(\mathbf{g} - \boldsymbol{\eta})_k = \sum_{j \in \mathbb{Z}^d} b_k(j)\, \psi_r(\,\cdot\,/h - \xi_j), \qquad \forall k \in \mathbb{Z}^d. \tag{5.13}$$

Recall that in order to show that $\mathbf{g} - \boldsymbol{\eta} \in \mathcal{N}$, we must show that $\|\mathbf{g} - \boldsymbol{\eta}\|_{\mathcal{N}} < \infty$ and additionally that for all compact $K \subset \mathbb{R}^d$, $\sum_{k \in \mathbb{Z}^d} \|(\mathbf{g} - \boldsymbol{\eta})_k\|_{L_\infty(K)} < \infty$. For the latter, let $K \subset \mathbb{R}^d$ be compact. Then

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}^d} \|(\mathbf{g} - \boldsymbol{\eta})_k\|_{L_\infty(K)} \\
&\quad \leqslant \sum_{k \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} |b_k(j)|\, \|\psi_r(\,\cdot\,/h - \xi_j)\|_{L_\infty(K)}, \qquad \text{by (5.13)}, \\
&\quad = \sum_{j \in \mathbb{Z}^d} \|\psi_r(\,\cdot\,/h - \xi_j)\|_{L_\infty(K)} \sum_{k \in \mathbb{Z}^d} |b_k(j)|, \qquad \text{by Fubini's Theorem}, \\
&\quad \leqslant \operatorname{const}(K, h) \left(\sum_{j \in \mathbb{Z}^d} \|\psi_r\|_{L_\infty(j+Q)} \right) \sup_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |b_k(j)|. \qquad (5.14)
\end{aligned}$$

Now, if $j \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, then

$$\begin{aligned}
\sum_{|k| \leqslant n} |b_k(j)| &\leqslant \left\| \sum_{|k| \leqslant n} \operatorname{signum}(\overline{b_k(j)})\, b_k \right\|_{\ell_\infty} = \left\| \Lambda \left(\sum_{|k| \leqslant n} \operatorname{signum}(\overline{b_k(j)})\, a_k \right) \right\|_{\ell_\infty} \\
&\leqslant \operatorname{const}(d, \zeta, \delta_0) \left\| \sum_{|k| \leqslant n} \operatorname{signum}(\overline{b_k(j)})\, a_k \right\|_{\ell_\infty}, \quad \text{by Lemma 4.3 (3)}, \\
&\leqslant \operatorname{const}(d, \zeta, \delta_0) \sup_{\ell \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |a_k(\ell)|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sup_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |b_k(j)| &\leqslant \operatorname{const}(d, \zeta, \delta_0) \sup_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |a_k(j)| \\
&= \operatorname{const}(d, \zeta, \delta_0) \sup_{j \in \mathbb{Z}^d} h^{-d}\, \|\tau_{r,h}(\,\cdot\,/h + j)\|_{\ell_1} \\
&\leqslant \operatorname{const}(d, \zeta, \delta_0)\, h^{-d}\, \|\tau_{r,h}(\,\cdot\,/h)\|_{L_1}, \qquad \text{by Lemma 4.1}, \\
&= \operatorname{const}(d, \zeta, \delta_0)\, \|\tau_{r,h}\|_{L_1}. \qquad (5.15)
\end{aligned}$$

Combining (5.15) and (5.14) yields $\sum_{k \in \mathbb{Z}^d} \|(\mathbf{g} - \boldsymbol{\eta})_k\|_{L_\infty(K)} < \infty$. Next we estimate $\|\mathbf{g} - \boldsymbol{\eta}\|_{\mathcal{N}}$. If $k \in \mathbb{Z}^d$, then

$$\begin{aligned}\|(\mathbf{g}-\boldsymbol{\eta})_k\|_{L_1} &\leqslant \sum_{j\in\mathbb{Z}^d} |b_k(j)|\ \|\psi_r(\cdot/h-\xi_j)\|_{L_1} = h^d\,\|\psi_r\|_{L_1}\,\|b_k\|_{\ell_1}\\ &\leqslant h^d\,\|\psi_r\|_{L_1}\,\mathrm{const}(d,\zeta,\delta_0)\,\|a_k\|_{\ell_1}, \qquad \text{by Lemma 4.3 (1)},\\ &= \mathrm{const}(d,\zeta,\delta_0)\,\|\psi_r\|_{L_1}\,\|\tau_{r,h}(\cdot-k/h)\|_{\ell_1}\\ &\leqslant \mathrm{const}(d,\zeta,\delta_0)\,\|\psi_r\|_{L_1}\,\|\tau_{r,h}\|_{L_1}, \qquad \text{by Lemma 4.1},\\ &\leqslant \mathrm{const}(d,\zeta,\delta_0)\,\|\tau_{r,h}\|_{L_1}\sum_{j\in\mathbb{Z}^d}\|\psi_r\|_{L_\infty(j+Q)}.\end{aligned}$$

Hence, $\sup_{k\in\mathbb{Z}^d}\|(\mathbf{g}-\boldsymbol{\eta})_k\|_{L_1}\leqslant \mathrm{const}(d,\zeta,\delta_0)\ \|\tau_{r,h}\|_{L_1}\sum_{j\in\mathbb{Z}^d}\|\psi_r\|_{L_\infty(j+Q)}$. On the other hand,

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{Z}^d}|(\mathbf{g}-\boldsymbol{\eta})_k|\Big\|_{L_\infty} &\leqslant \sup_{x\in\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}|b_k(j)|\ |\psi_r(x/h-\xi_j)|\\ &= \sup_{x\in\mathbb{R}^d}\sum_{j\in\mathbb{Z}^d}|\psi_r(x/h-\xi_j)|\sum_{k\in\mathbb{Z}^d}|b_k(j)|,\\ &\qquad \text{by Fubini's theorem},\\ &\leqslant \Big\|\sum_{j\in\mathbb{Z}^d}|\psi_r(\cdot-\xi_j)|\Big\|_{L_\infty}\sup_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|b_k(j)|\\ &\leqslant \mathrm{const}(d,\zeta,\delta_0)\sum_{j\in\mathbb{Z}^d}\|\psi_r\|_{L_\infty(j+Q)}\,\|\tau_{r,h}\|_{L_1}, \qquad \text{by (5.15)}.\end{aligned}$$

Therefore, $\|\mathbf{g}-\boldsymbol{\eta}\|_{\mathcal{N}}\leqslant \mathrm{const}(d,\zeta,\delta_0)\ \|\tau_{r,h}\|_{L_1}\sum_{j\in\mathbb{Z}^d}\|\psi_r\|_{L_\infty(j+Q)}$. In particular, $\mathbf{g}-\boldsymbol{\eta}\in\mathcal{N}$, and since $\boldsymbol{\eta}\in\mathcal{N}$, it follows that $\mathbf{g}\in\mathcal{N}$; hence, $\mathbf{g}\in S^h(\phi_r;\Xi)^{\mathbb{Z}^d}\cap\mathcal{N}$, and so we conclude that

$$\mathrm{dist}(\boldsymbol{\eta},(S^h(\phi_r;\Xi))^{\mathbb{Z}^d}\cap\mathcal{N};\mathcal{N})\leqslant \mathrm{const}(d,\zeta,\delta_0)\ \|\tau_{r,h}\|_{L_1}\sum_{j\in\mathbb{Z}^d}\|\psi_r\|_{L_\infty(j+Q)}.$$

Taking the infimum over all appropriate ζ completes the proof. ∎

6. PROOF OF THEOREM 3.1

The following string of lemmata will be used to prove Theorem 3.1 at the end of this section.

LEMMA 6.1. *Let* $0 \leqslant a < b \leqslant \infty$, *and let* $F \in C^m(a, b)$ *for some* $m \in \mathbb{N}$. *Then there exist* $p_{\alpha,k} \in C^\infty(\mathbb{R}^d \backslash 0)$, $1 \leqslant k \leqslant |\alpha| \leqslant m$, *such that* $p_{\alpha,k}$ *is homogeneous of degree* $k - |\alpha|$ *and*

$$D^\alpha(F(|\cdot|)) = \sum_{k=1}^{|\alpha|} p_{\alpha,k} F^{(k)}(|\cdot|) \tag{6.2}$$

on $\Omega := \{x \in \mathbb{R}^d : a < |x| < b\}$ *for all* $1 \leqslant |\alpha| \leqslant m$.

Proof. If $|\alpha| = 1$, then $D^\alpha(F(|\cdot|)) = F'(|\cdot|)\, D^\alpha|\cdot|$ which settles the case $m = 1$ since $p_{\alpha,1} := D^\alpha\,|\cdot| \in C^\infty(\mathbb{R}^d \backslash 0)$ and is homogeneous of degree 0. Proceeding by induction on m, assume that (6.2) holds for all $1 \leqslant |\alpha| \leqslant m-1$ and consider m. Let $|\alpha| = m-1$ and $|\beta| = 1$. Then

$$\begin{aligned} D^{\alpha+\beta}(F(|\cdot|)) &= D^\beta\left(\sum_{k=1}^{|\alpha|} p_{\alpha,k} F^{(k)}(|\cdot|)\right), \qquad \text{by the induction hypothesis,} \\ &= \sum_{k=1}^{|\alpha|} ((D^\beta p_{\alpha,k})\, F^{(k)}(|\cdot|) + p_{\alpha,k} F^{(k+1)}(|\cdot|)\, D^\beta(|\cdot|)), \\ &\qquad \text{since} \quad |\beta| = 1, \\ &= \sum_{k=1}^{|\alpha|} (D^\beta p_{\alpha,k})\, F^{(k)}(|\cdot|) + \sum_{k=2}^{|\alpha|+1} p_{\alpha,k-1}(D^\beta\,|\cdot|)\, F^{(k)}(|\cdot|). \end{aligned}$$

Noting that both $D^\beta p_{\alpha,k}$ and $p_{\alpha,k-1}\,(D^\beta\,|\cdot|)$ are in $C^\infty(\mathbb{R}^d \backslash 0)$ and homogeneous of degree $k - |\alpha + \beta|$, we complete the induction. ∎

LEMMA 6.3. *Let* $n \geqslant d$, $\varepsilon \in (0, 1)$, *and* $\delta \in (0, \infty)$. *Let* $F \in C[0, \delta) \cap C^n(0, \delta)$. *If* $v \in \mathcal{D}(\delta B)$, *then*

$$\begin{aligned} &\|(1 + |\cdot|)^{n-d+\varepsilon/2}\,(vF(|\cdot|))^\vee\|_{L_1} \\ &\qquad \leqslant \operatorname{const}(d, n, \delta, \varepsilon, v)\left(\sup_{0<\rho<\delta} |F(\rho)| + \max_{1 \leqslant k \leqslant n}\ \sup_{0<\rho<\delta} \frac{|F^{(k)}(\rho)|}{\rho^{n-d+\varepsilon-k}}\right). \end{aligned}$$

Proof. Without loss of generality assume that the right side of our inequality is finite. Let $v \in \mathcal{D}(\delta B)$, and let $q \in (1, 2]$ be the middling value satisfying $\varepsilon > d - d/q > \varepsilon/2$. Put $\tau := (vF(|\cdot|))^\vee$, and let p be the exponent conjugate to q. Then

$$
\begin{aligned}
&\|(1+|\cdot|)^{n-d+\varepsilon/2}\,\tau\|_{L_1}\\
&\quad\leqslant \operatorname{const}(d,n,\varepsilon)\sum_{j\in\mathbb{Z}^d}(1+|j|)^{-d+\varepsilon/2}\,\|(1+|\cdot|)^n\,\tau\|_{L_1(j+Q)}\\
&\quad\leqslant \operatorname{const}(d,n,\varepsilon)\sum_{j\in\mathbb{Z}^d}(1+|j|)^{-d+\varepsilon/2}\,\|(1+|\cdot|)^n\,\tau\|_{L_p(j+Q)}\\
&\quad\leqslant \operatorname{const}(d,n,\varepsilon)\left(\sum_{j\in\mathbb{Z}^d}(1+|j|)^{(-d+\varepsilon/2)\,q}\right)^{1/q}\|(1+|\cdot|)^n\,\tau\|_{L_p},
\end{aligned}
$$

by Hölder's inequality.

Note that $q(-d+\varepsilon/2)<-d$ follows from the assumption $d-d/q>\varepsilon/2$. Therefore,

$$
\begin{aligned}
\|(1+|\cdot|)^{n-d+\varepsilon/2}\,\tau\|_{L_1} &\leqslant \operatorname{const}(d,n,\varepsilon)\,\|(1+|\cdot|)^n\,\tau\|_{L_p}\\
&\leqslant \operatorname{const}(d,n,\varepsilon)\,\|\hat{\tau}\|_{W_q^n(\mathbb{R}^d)},
\end{aligned}
\tag{6.4}
$$

by the (extended) Hausdorff–Young Theorem. Put $\Omega := \operatorname{supp} v$. Then

$$
\begin{aligned}
&\|\hat{\tau}\|_{W_q^n(\mathbb{R}^d\backslash 0)}\\
&\leqslant \operatorname{const}(d,n,\varepsilon,v)\,\|F(|\cdot|)\|_{W_q^n(\Omega\backslash 0)}\\
&\leqslant \operatorname{const}(d,n,\varepsilon,v)\max_{|\alpha|\leqslant n}\|D^\alpha(F(|\cdot|))\|_{L_q(\Omega\backslash 0)}\\
&\leqslant \operatorname{const}(d,n,\varepsilon,v)\left(\|F(|\cdot|)\|_{L_q(\Omega\backslash 0)}+\max_{1\leqslant|\alpha|\leqslant n}\sum_{k=1}^{|\alpha|}\||\cdot|^{k-|\alpha|}\,F^{(k)}(|\cdot|)\|_{L_q(\Omega\backslash 0)}\right),
\end{aligned}
$$

by Lemma 6.1,

$$
\begin{aligned}
&\leqslant \operatorname{const}(d,n,\varepsilon,v)\left(\sup_{0<\rho<\delta}|F(\rho)|+\max_{1\leqslant k\leqslant n}\||\cdot|^{k-n}\,F^{(k)}(|\cdot|)\|_{L_q(\Omega\backslash 0)}\right)\\
&\leqslant \operatorname{const}(d,n,\varepsilon,v)\left(\sup_{0<\rho<\delta}|F(\rho)|+\||\cdot|^{\varepsilon-d}\|_{L_q(\Omega\backslash 0)}\right.\\
&\qquad\left.\times\max_{1\leqslant k\leqslant n}\||\cdot|^{k-n+d-\varepsilon}\,F^{(k)}(|\cdot|)\|_{L_\infty(\delta B\backslash 0)}\right)\\
&\leqslant \operatorname{const}(d,n,\delta,\varepsilon,v)\left(\sup_{0<\rho<\delta}|F(\rho)|+\max_{1\leqslant k\leqslant n}\sup_{0<\rho<\delta}\frac{|F^{(k)}(\rho)|}{\rho^{n-d+\varepsilon-k}}\right)
\end{aligned}
$$

as $q(\varepsilon-d)>-d$ is implied by $\varepsilon>d-d/q$. So with (6.4) in view, in order to complete the proof of the lemma, we need only show that $D^\alpha\hat{\tau}\in L_q$ for all $|\alpha| \leqslant n$. Since $D^\alpha \hat{\tau} \in L_q(\mathbb{R}^d \backslash 0)$ has been established, it suffices to show that

$$\langle g, D^\alpha \hat{\tau} \rangle = \int_{\mathbb{R}^d \backslash 0} g D^\alpha \hat{\tau} \, dm, \tag{6.5}$$

for all $g \in \mathscr{D}$, $|\alpha| \leqslant n$. So let $g \in \mathscr{D}$, $|\alpha| \leqslant n$. Since $F \in C([0, \delta))$, (6.5) holds if $\alpha = 0$; so assume $|\alpha| > 0$. By Lemma 6.1,

$$\begin{aligned} |D^\alpha(F(|\cdot|))| &= \left| \sum_{k=1}^{|\alpha|} p_{\alpha, k} F^{(k)}(|\cdot|) \right| \\ &\leqslant \operatorname{const}(d, n, \varepsilon, F) \sum_{k=1}^{|\alpha|} |\cdot|^{k-|\alpha|} \, |\cdot|^{\varepsilon+n-d-k} \\ &\leqslant \operatorname{const}(d, n, \varepsilon, F) \, |\cdot|^{\varepsilon+n-d-|\alpha|}. \end{aligned}$$

Thus $F(|\cdot|) \in C(\mathbb{R}^d) \cap C^{n-d}(\mathbb{R}^d \backslash 0)$ and the restriction of $D^\alpha(F(|\cdot|))$ to $\mathbb{R}^d \backslash 0$ admits a continuous extension to all of $\mathbb{R}^d$ for all $|\alpha| \leqslant n-d$. It follows that $F(|\cdot|) \in C^{n-d}(\mathbb{R}^d)$. Consequently, $\hat{\tau} = vF(|\cdot|) \in C^{n-d}(\mathbb{R}^d)$ and (6.5) holds whenever $|\alpha| \leqslant n-d$. So assume $n-d < |\alpha| \leqslant n$. Let $p \in \Pi_{n-d}$ be the Taylor approximation to $\hat{\tau}$ (at 0). Let $\sigma \in \mathscr{D}(B)$ be identically 1 on a neighborhood of 0, and define $\sigma_\ell := \sigma(\ell\, \cdot)$, $\ell \in \mathbb{N}$. Then

$$\langle g, D^\alpha \hat{\tau} \rangle = \langle \sigma_\ell g, D^\alpha \hat{\tau} \rangle + \langle (1-\sigma_\ell)\, g, D^\alpha \hat{\tau} \rangle.$$

Since $(1-\sigma_\ell)\, g \in \mathscr{D}(\mathbb{R}^d \backslash 0)$ and $\hat{\tau} \in C^n(\mathbb{R}^d \backslash 0)$, we have

$$\langle (1-\sigma_\ell)\, g, D^\alpha \hat{\tau} \rangle = \int_{\mathbb{R}^d \backslash 0} (1-\sigma_\ell)\, g D^\alpha \hat{\tau} \, dm \to \int_{\mathbb{R}^d \backslash 0} g D^\alpha \hat{\tau} \, dm \qquad \text{as } \ell \to \infty.$$

Thus, in order to establish (6.5), it suffices to show that $\langle \sigma_\ell g, D^\alpha \hat{\tau} \rangle \to 0$ as $\ell \to \infty$. Since $|\alpha| > n-d$, we have $D^\alpha p = 0$. Hence

$$\begin{aligned} |\langle \sigma_\ell g, D^\alpha \hat{\tau} \rangle| &= |\langle \sigma_\ell g, D^\alpha(\hat{\tau} - p) \rangle| = |\langle D^\alpha(\sigma_\ell g), \hat{\tau} - p \rangle| \\ &\leqslant \|D^\alpha(\sigma_\ell g)\|_{L_\infty} \, \|\hat{\tau} - p\|_{L_\infty(B/\ell)} \, m(B/\ell) \\ &= O(\ell^{|\alpha|}) \, o(\ell^{-(n-d)}) \, O(\ell^{-d}) = o(1) \end{aligned}$$

as $\ell \to 0$. ∎

LEMMA 6.6. *Let* $\varepsilon \in (0, 1)$, $\delta \in (0, \infty)$. *Let* $G \in C[0, \delta) \cap C^d(0, \delta)$ *satisfy* $G \neq 0$ *on all of* $[0, \delta)$. *If* $v \in \mathscr{D}(\delta B)$, *then*

$$\left\| (1+|\cdot|)^{\varepsilon/2} \left(\frac{v}{G(|\cdot|)} \right)^{\vee} \right\|_{L_1} \leqslant \operatorname{const}(d, \delta, \varepsilon, v) \left(1 + \max_{1 \leqslant k \leqslant d} \sup_{0<\rho<\delta} \left| \frac{G^{(k)}(\rho)}{G(\rho)\, \rho^{\varepsilon - k}} \right| \right)^d \sup_{0<\rho<\delta} \frac{1}{|G(\rho)|}.$$

Proof. Put $F(\rho) := 1/G(\rho)$, $0 \leqslant \rho < \delta$. Then $F \in C[0, \delta) \cap C^d(0, \delta)$, and so in view of Lemma 6.3, in order to prove our lemma, it suffices to show that

$$\max_{1 \leqslant k \leqslant d} \sup_{0<\rho<\delta} \frac{|F^{(k)}(\rho)|}{\rho^{\varepsilon - k}} \leqslant \operatorname{const}(d, \delta, \varepsilon, v) \left(1 + \max_{1 \leqslant k \leqslant d} \sup_{0<\rho<\delta} \left| \frac{G^{(k)}(\rho)}{G(\rho)\, \rho^{\varepsilon - k}} \right| \right)^d \sup_{0<\rho<\delta} \frac{1}{|G(\rho)|}.$$

For this it suffices to prove that for all $1 \leqslant k \leqslant d$,

$$|G(\rho)\, F^{(k)}(\rho)| \leqslant \operatorname{const}(d, \delta, \varepsilon, v) \left(1 + \max_{1 \leqslant j \leqslant k} \sup_{0<\rho<\delta} \left| \frac{G^{(j)}(\rho)}{G(\rho)\, \rho^{\varepsilon - j}} \right| \right)^k \rho^{\varepsilon - k}, \quad 0 < \rho < \delta. \tag{6.7}$$

Differentiating the identity $F(\rho)\, G(\rho) = 1$ and solving for $G(\rho)\, F^{(k)}(\rho)$ yields

$$G(\rho)\, F^{(k)}(\rho) = -\sum_{j=0}^{k-1} \binom{k}{j} F^{(j)}(\rho)\, G^{(k-j)}(\rho), \qquad 0 < \rho < \delta, \quad 1 \leqslant k \leqslant d. \tag{6.8}$$

For $k = 1$ this reads $G(\rho)\, F'(\rho) = -G'(\rho)/G(\rho)$ which proves (6.7) for the case $k = 1$. Proceeding by induction, assume that (6.7) holds for all k, $1 \leqslant k \leqslant k' < d$, and consider $k = k' + 1$. Let $\rho \in (0, \delta)$. In view of (6.8), in order to prove (6.7), it suffices to show that $|F^{(j)}(\rho)\, G^{(k-j)}(\rho)|$ is bounded by the right side of (6.7) for all $j = 0, 1, ..., k-1$. For $j = 0$ we have $|F(\rho)\, G^{(k)}(\rho)| = |G^{(k)}(\rho)/(G(\rho)\, \rho^{\varepsilon - k})|\, \rho^{\varepsilon - k}$ which is bounded by the right side of (6.7). For $1 \leqslant j \leqslant k-1$, we employ the induction hypothesis to write

$$|F^{(j)}(\rho)\, G^{(k-j)}(\rho)|$$
$$\leqslant \operatorname{const}(d, \delta, \varepsilon, \nu) \left(1 + \max_{1 \leqslant \ell \leqslant j} \sup_{0<\rho<\delta} \left|\frac{G^{(\ell)}(\rho)}{G(\rho)\, \rho^{\varepsilon-\ell}}\right|\right)^j \left|\frac{\rho^{\varepsilon-j} G^{(k-j)}(\rho)}{G(\rho)}\right|$$
$$= \operatorname{const}(d, \delta, \varepsilon, \nu) \left(1 + \max_{1 \leqslant \ell \leqslant j} \sup_{0<\rho<\delta} \left|\frac{G^{(\ell)}(\rho)}{G(\rho)\, \rho^{\varepsilon-\ell}}\right|\right)^j \left|\frac{G^{(k-j)}(\rho)}{G(\rho)\, \rho^{\varepsilon-k+j}}\right| \rho^{2\varepsilon-k}$$

which is bounded by the right side of (6.7). ∎

LEMMA 6.9. *Let $\varepsilon \in (0, 1)$ and $\delta \in (0, \infty)$. Let $F \in C^{d+1}((\delta, \infty))$. If $\sigma \in \mathscr{D}$ satisfies $\sigma = 1$ on δB, then*

$$\sum_{j \in \mathbb{Z}^d} \|((1-\sigma)\, F(|\cdot|))^\vee\|_{L_\infty(j+Q)} \leqslant \operatorname{const}(d, \sigma, \delta, \varepsilon) \max_{0 \leqslant k \leqslant d+1} \sup_{\delta<\rho<\infty} \frac{|F^{(k)}(\rho)|}{\rho^{-d-\varepsilon}}.$$

Proof. First note that

$$\sum_{j \in \mathbb{Z}^d} \|((1-\sigma)\, F(|\cdot|))^\vee\|_{L_\infty(j+Q)} \leqslant \operatorname{const}(d)\, \|(1+|\cdot|)^{d+1}\, ((1-\sigma)\, F(|\cdot|)^\vee\|_{L_\infty}$$
$$\leqslant \operatorname{const}(d)\, \|(1-\sigma)\, F(|\cdot|)\|_{W_1^{d+1}(\mathbb{R}^d)},$$

by (extended) Hausdorff–Young Theorem,

$$\leqslant \operatorname{const}(d, \sigma)\, \|F(|\cdot|)\|_{W_1^{d+1}(\mathbb{R}^d \setminus \delta B)}.$$

Since the functions $p_{\alpha, k}$ are homogeneous of degree $\leqslant 0$, it follows from Lemma 6.1 that

$$\|F(|\cdot|)\|_{W_1^{d+1}(\mathbb{R}^d \setminus \delta B)} \leqslant \operatorname{const}(d, \delta) \max_{0 \leqslant k \leqslant d+1} \|F^{(k)}(|\cdot|)\|_{L_1(\mathbb{R}^d \setminus \delta B)}$$
$$= \operatorname{const}(d, \delta) \max_{0 \leqslant k \leqslant d+1} \int_\delta^\infty |F^{(k)}(\rho)|\, \rho^{d-1}\, d\rho$$
$$\leqslant \operatorname{const}(d, \delta, \varepsilon) \max_{0 \leqslant k \leqslant d+1} \sup_{\delta<\rho<\infty} \frac{|F^{(k)}(\rho)|}{\rho^{-d-\varepsilon}}. \quad ∎$$

Proof of Theorem 3.1. In case $\gamma_0 > 0$, and with (ii) in view, we may assume without loss of generality that $m - d + \varepsilon < \gamma_0$. Note that if $\gamma_0 = 0$, then $m = d$. Put $\delta_1 := \inf\{t \geqslant 0 : \lambda(t) = 0\} \in (0, \infty]$.

CLAIM 6.10. *There exists $\mu \in (0, \gamma_0] \cup \{\gamma_0\}$ such that*

(1) $|\phi(x)| = o(|x|^{\gamma_0 - \mu})$ *as* $|x| \to \infty$;

(2) $(1+|\cdot|)^{\gamma_0-\mu}\, (\nu|\cdot|^{\gamma_0}/\lambda(|\cdot|))^\vee \in L_1$, $\forall \nu \in \mathscr{D}(\delta_1 B)$.

Proof. Let $v \in \mathscr{D}(\delta_1 B)$. There exists $\delta \in (0, \delta_1)$ such that $v \in \mathscr{D}(\delta B)$. Define $F(\rho) := \rho^{\gamma^0}/\lambda(\rho)$, $\rho \in [0, \delta]$. Note that $F \in C([0, \delta])$. That $F \in C^m((0, \delta])$ follows from (iii) and the fact that $\delta < \delta_1$. We will show that

$$(1+|\cdot|)^{m-d+\varepsilon/2} \left(\frac{v\,|\cdot|^{\gamma_0}}{\lambda(|\cdot|)} \right)^{\vee} \in L_1. \tag{6.11}$$

In view of Lemma 6.3 (with $n := m$), it suffices to show that

$$|F^{(k)}(\rho)| = O(\rho^{m-d+\varepsilon-k}) \qquad \text{as} \quad \rho \to 0, \tag{6.12}$$

for all $1 \leqslant k \leqslant m$. Differentiating the identity $\lambda(\rho)\, F(\rho) = \rho^{\gamma_0}$ and solving for $F^{(k)}(\rho)$ yields

$$F^{(k)}(\rho) = \frac{1}{\lambda(\rho)} \Bigg(\gamma_0(\gamma_0 - 1) \cdots (\gamma_0 - k + 1)\, \rho^{\gamma_0 - k} - \sum_{j=1}^{k} \binom{k}{j} \lambda^{(j)}(\rho)\, F^{(k-j)}(\rho) \Bigg), \tag{6.13}$$

$1 \leqslant k \leqslant m$, $0 < \rho < \delta$. Note that for $1 \leqslant k \leqslant m$,

$$|\gamma_0(\gamma_0 - 1) \cdots (\gamma_0 - k + 1)\, \rho^{\gamma_0 - k}| = O(\rho^{m-d+\varepsilon-k}) \qquad \text{as} \quad \rho \to 0,$$

since $m - d + \varepsilon < \gamma_0$ is assumed in case $\gamma_0 > 0$. That (6.12) holds in case $k = 1$ follows readily from (6.13), (iv), and the fact that $|F(\rho)| = O(\rho^{\gamma_0})$ as $\rho \to 0$. Proceeding by induction, assume that (6.12) holds for all k, $1 \leqslant k \leqslant k' < m$, and consider $k = k' + 1$. By (iv) and the induction hypothesis, it follows that

$$\begin{aligned} |\lambda^{(j)}(\rho)\, F^{(k-j)}(\rho)| &= O(\rho^{\varepsilon - j})\, O(\rho^{m-d+\varepsilon+j-k}) \\ &= O(\rho^{m-d+\varepsilon-k}) \qquad \text{as} \quad \rho \to 0, \end{aligned}$$

for all $1 \leqslant j \leqslant k-1$. As for $j = k$, we have by (iv) that

$$|\lambda^{(k)}(\rho)\, F(\rho)| = O(\rho^{\varepsilon - k})\, O(\rho^{\gamma_0}) = O(\rho^{m-d+\varepsilon-k}) \qquad \text{as} \quad \rho \to 0.$$

Therefore, in view of (6.13), estimate (6.12) holds for $k = k' + 1$, and thus (6.11) is proved.

Case 1. $\gamma_0 > 0$.

Since $\gamma_0 > \lceil \gamma_0 - \bar{\mu} \rceil$ (by (ii)), we must have $0 < \bar{\mu} \leqslant \gamma_0$. Hence $\varnothing \neq (0, \bar{\mu}) \subset (0, \gamma_0]$. Note that by definition of $\bar{\mu}$, condition (1) holds for all $\mu \in (0, \bar{\mu})$. On the other hand,

$$m - d + \varepsilon/2 = \lceil \gamma_0 - \bar{\mu} \rceil + \varepsilon/2 \geqslant \gamma_0 - \bar{\mu} + \varepsilon/2 > \gamma_0 - \mu$$

for $\mu \in (0, \bar{\mu})$ sufficiently close to $\bar{\mu}$. Hence, by (6.11), condition (2) holds for some $\mu \in (0, \bar{\mu})$.

Case 2. $\gamma_0 = 0$.

With $\mu := 0$, condition (1) follows from (i). In particular, $\bar{\mu} = 0$. Hence $m - d + \varepsilon/2 = \varepsilon/2$ and thus condition (2) is a consequence of (6.11). Hence the claim.

Let $\delta \in (0, \pi)$ be such that $\lambda \neq 0$ on all of $[0, \delta]$. Let $\hat{\eta} \in \mathscr{D}(\delta B)$ satisfy $\hat{\eta} = 1$ on $\frac{1}{2}\delta B$. Let $\sigma \in \mathscr{D}(\pi B)$ satisfy $\sigma = 1$ on δB.

CLAIM 6.14. *If* $G \in C^{d+1}(\delta, \infty)$, *then*

$$\sum_{j \in \mathbb{Z}^d} \|((1-\sigma)\,|\cdot|^{-\gamma_0}\, G(|\cdot|))^{\vee})\|_{L_\infty(j+Q)}$$

$$\leqslant \mathrm{const}(d, \sigma, \delta, \varepsilon, \gamma_0) \max_{0 \leqslant k \leqslant d+1} \sup_{\delta < \rho < \infty} \frac{|G^{(k)}(\rho)|}{\rho^{\gamma_0 - d - \varepsilon}}.$$

Proof. Let $G \in C^{d+1}(\delta, \infty)$ and put $F(\rho) := \rho^{-\gamma_0} G(\rho)$, $\rho > 0$. In view of Lemma 6.9, it suffices to show that

$$\sup_{\delta < \rho < \infty} \frac{|F^{(k)}(\rho)|}{\rho^{-d-\varepsilon}} \leqslant \mathrm{const}(d, \delta, \varepsilon, \gamma_0) \max_{0 \leqslant j \leqslant d+1} \sup_{\delta < \rho < \infty} \frac{|G^{(j)}(\rho)|}{\rho^{\gamma_0 - d - \varepsilon}},$$

for all $0 \leqslant k \leqslant d+1$. That this is true can be seen by noting that for $0 \leqslant k \leqslant d+1$ and $\delta < \rho < \infty$,

$$F^{(k)}(\rho) = \sum_{j=0}^{k} \binom{k}{j} (-\gamma_0)(-\gamma_0 - 1) \cdots (-\gamma_0 - (k-j-1))\, \rho^{-\gamma_0 - (k-j)} G^{(j)}(\rho).$$

Hence the claim.

CLAIM 6.15. *The stationary ladder* $(S^h(\phi; \Xi))_h$ *provides* L_p*-approximation of order* γ_0 *for all* $1 \leqslant p \leqslant \infty$ *whenever* Ξ *is a sufficiently small perturbation of* $\mathbb{Z}^d$.

Proof. In order to apply Theorem 5.8, put $\phi_h := \phi$, $h > 0$; then $\lambda_h = \lambda(|\cdot|)$, $h > 0$. It follows from Claim 6.10 (with $v := \hat{\eta}(\cdot/h)$) that there exists $\mu \in (0, \gamma_0] \cup \{\gamma_0\}$ such that conditions (i) and (ii) of Theorem 5.8 hold. Hence, in view of Theorem 5.8, in order to prove the claim it suffices to show that

$$\sum_{j \in \mathbb{Z}^d} \|((1-\sigma)\,|\cdot|^{-\gamma_0}\, \lambda(|\cdot|))^{\vee}\|_{L_\infty(j+Q)} < \infty, \tag{6.16}$$

and

$$\left\| \left(\frac{\hat{\eta}(\,\cdot\,/h)\,|\cdot|^{\gamma_0}}{\lambda(|\cdot|)} \right)^{\vee} \right\|_{L_1} = O(h^{\gamma_0}) \qquad \text{as} \quad h \to 0. \tag{6.17}$$

That (6.16) holds follows from (iii), (v), and Claim 6.14 (with $G := \lambda$). So, we now consider (6.17). If $h < 1/2$, then

$$\begin{aligned}
\left\| \left(\frac{\hat{\eta}(\,\cdot\,/h)\,|\cdot|^{\gamma_0}}{\lambda(|\cdot|)} \right)^{\vee} \right\|_{L_1} &= h^{\gamma_0} \left\| \left(\frac{\hat{\eta}\,|\cdot|^{\gamma_0}}{\lambda(|h\cdot|)} \right)^{\vee} \right\|_{L_1} = h^{\gamma_0} \left\| \left(\hat{\eta}\,|\cdot|^{\gamma_0} \frac{\hat{\eta}(h\,\cdot\,)}{\lambda(|h\cdot|)} \right)^{\vee} \right\|_{L_1} \\
&\leqslant h^{\gamma_0} \, \|(\hat{\eta}\,|\cdot|^{\gamma_0})^{\vee}\|_{L_1} \left\| \left(\frac{\hat{\eta}(h\,\cdot\,)}{\lambda(|h\cdot|)} \right)^{\vee} \right\|_{L_1} \\
&= h^{\gamma_0} \, \|(\hat{\eta}\,|\cdot|^{\gamma_0})^{\vee}\|_{L_1} \left\| \left(\frac{\hat{\eta}}{\lambda(|\cdot|)} \right)^{\vee} \right\|_{L_1}.
\end{aligned}$$

That $(\hat{\eta}\,|\cdot|^{\gamma_0})^{\vee} \in L_1$ is an easy consequence of Lemma 6.3 while $(\hat{\eta}/\lambda(|\cdot|))^{\vee} \in L_1$ follows from (iii), (iv), and Lemma 6.6 (with $v := \hat{\eta}$ and $G := \lambda$). Therefore (6.17) holds and the claim is proved.

Having dispensed with the stationary case, we turn now to the non-stationary half of the theorem. Assume that there exists θ, a, $N \in (0, \infty)$ such that (vi) and (vii) hold. Let κ: $(0, 1] \to (0, \infty)$ satisfy

$$\limsup_{h \to 0} \kappa(h)^{\theta} \log(1/h) < \frac{\pi^{\theta}}{\gamma_1}, \qquad \text{for some} \quad \gamma_1 \in (0, \infty), \tag{6.18}$$

and define $\phi_h := \phi(\kappa(h)\,\cdot\,)$, $h \in (0, 1]$. Since $\kappa(h) \to 0$ as $h \to 0$, we may assume without loss of generality that $\kappa(h) \leqslant 1$ $\forall h \in (0, 1]$. Note that $\hat{\phi}_h = \kappa(h)^{-d}\,\hat{\phi}(\,\cdot\,/\kappa(h))$ and so $\mathbb{R}^d \backslash 0$, $\hat{\phi}_h$ can be identified with $|\cdot|^{-\gamma_0}\,\kappa(h)^{\gamma_0 - d}\,\lambda(|\cdot|/\kappa(h))$. So in the terminology of Theorem 5.8, $\lambda_h = \kappa(h)^{\gamma_0 - d}\lambda(|\cdot|/\kappa(h))$, $h \in (0, 1]$. By (vi), $\lambda \neq 0$ on all of $[0, \infty)$ and hence it follows from Claim 6.10 that there exists $\mu \in (0, \gamma_0] \cup \{\gamma_0\}$ such that (i) and (ii) of Theorem 5.8 are satisfied. For $0 < r \leqslant h \leqslant 1$, put

$$\Gamma(r, h) := \left\| \left(\frac{\hat{\eta}(\,\cdot\,/h)\,|\cdot|^{\gamma_0}}{\lambda_r} \right)^{\vee} \right\|_{L_1} \sum_{j \in \mathbb{Z}^d} \|((1 - \sigma)\,|\cdot|^{-\gamma_0}\,\lambda_r)^{\vee}\|_{L_\infty(j + Q)}.$$

Then, in view of Theorem 5.8, in order to complete the proof of our theorem, it suffices to show that

$$\sup_{0 < r \leqslant h} \Gamma(r, h) = O(h^{\gamma_0 + \gamma_1}) \qquad \text{as} \quad h \to 0. \tag{6.19}$$

Note that for all $0<r\leqslant h\leqslant 1$,

$$\Gamma(r,h)=h^{\gamma_0}\left\|\left(\frac{\hat{\eta}\,|\cdot|^{\gamma_0}}{\lambda(h\,|\cdot|/\kappa(r))}\right)^{\vee}\right\|_{L_1}\sum_{j\in\mathbb{Z}^d}\|((1-\sigma)\,|\cdot|^{-\gamma_0}\,\lambda(|\cdot|/\kappa(r)))^{\vee}\|_{L_\infty(j+Q)}. \tag{6.20}$$

By (iii), (iv), (vi), and (vii),

$$C_1:=\sup_{0<\rho<\infty}\frac{\exp(-a\rho^\theta)}{|\lambda(\rho)|}<\infty;$$

$$C_2:=\max_{0\leqslant k\leqslant d+1}\ \sup_{\delta<\rho<\infty}\frac{|\lambda^{(k)}(\rho)|}{\rho^N\exp(-\rho^\theta)}<\infty;$$

$$C_3:=\max_{1\leqslant k\leqslant d}\ \sup_{0<\rho<\infty}\frac{|\lambda^{(k)}(\rho)|}{\rho^{\varepsilon-k}}<\infty.$$

CLAIM 6.21. *For all* $0<r\leqslant h\leqslant 1$,

$$\left\|\left(\frac{\hat{\eta}\,|\cdot|^{\gamma_0}}{\lambda(h\,|\cdot|/\kappa(r))}\right)^{\vee}\right\|_{L_1}$$
$$\leqslant \mathrm{const}(d,\gamma_0,\delta,\varepsilon,\eta,C_1,C_3)\,\kappa(r)^{-d}\exp(a(d+1)(h\delta/\kappa(r))^\theta).$$

Proof. First of all,

$$\left\|\left(\frac{\hat{\eta}\,|\cdot|^{\gamma_0}}{\lambda(h\,|\cdot|/\kappa(r))}\right)^{\vee}\right\|_{L_1}\leqslant\left\|\left(\frac{\hat{\eta}}{\lambda(h\,|\cdot|/\kappa(r))}\right)^{\vee}\right\|_{L_1}\|(\hat{\eta}(\,\cdot\,/2)\,|\cdot|^{\gamma_0})^{\vee}\|_{L_1}. \tag{6.22}$$

Note that, with $v:=\hat{\eta}$ and $G:=\lambda(h\cdot/\kappa(r))$, the hypothesis of Lemma 6.6 is satisfied. Now,

$$\max_{1\leqslant k\leqslant d}\ \sup_{0<\rho<\delta}\frac{|G^{(k)}(\rho)|}{\rho^{\varepsilon-k}}=(h/\kappa(r))^\varepsilon\max_{1\leqslant k\leqslant d}\ \sup_{0<\rho<\delta}\frac{|\lambda^{(k)}(h\rho/\kappa(r))|}{(h\rho/\kappa(r))^{\varepsilon-k}}\leqslant C_3(h/\kappa(r))^\varepsilon. \tag{6.23}$$

On the other hand,

$$\sup_{0<\rho<\delta}\frac{1}{|G(\rho)|}=\sup_{0<\rho<\delta}\frac{\exp(-a(h\rho/\kappa(r))^\theta)}{\exp(-a(h\rho/\kappa(r))^\theta)\,\lambda(h\rho/\kappa(r))}\leqslant C_1\exp(a(h\delta/\kappa(r))^\theta). \tag{6.24}$$

It follows from (6.23) and (6.24) that

$$\left(1+\max_{1\leqslant k\leqslant d}\ \sup_{0<\rho<\delta}\left|\frac{G^{(k)}(\rho)}{G(\rho)\,\rho^{\varepsilon-k}}\right|\right)^{d}\sup_{0<\rho<\delta}\frac{1}{|G(\rho)|}$$
$$\leqslant \mathrm{const}(d, C_1, C_3)(1+(h/\kappa(r))^{\varepsilon})^{d}\exp(a(d+1)(h\delta/\kappa(r))^{\theta})$$
$$\leqslant \mathrm{const}(d, C_1, C_3)\,\kappa(r)^{-d}\exp(a(d+1)(h\delta/\kappa(r))^{\theta}),$$

for all $0<r\leqslant h\leqslant 1$. In view of (6.22) and Lemma 6.6, the claim is proved.

CLAIM 6.25. *There exists $h_1\in(0,1]$ such that*

$$\sum_{j\in\mathbb{Z}^d}\|((1-\sigma)\,|\cdot|^{-\gamma_0}\,\lambda(|\cdot/\kappa(r)|))^{\vee}\|_{L_\infty(j+Q)}$$
$$\leqslant C_2\,\mathrm{const}(d,\sigma,\delta,N,\varepsilon,\gamma_0)\,\kappa(r)^{-d-1-N}\exp(-\kappa(r)^{-\theta}\,\delta^{\theta}),$$
$$\forall 0<r\leqslant h_1.$$

Proof. Put $G:=\lambda(\cdot/\kappa(r))\in C^{d+1}((\delta,\infty))$. In view of Claim 6.14, it suffices to show that there exists $h_1\in(0,1]$ such that for all $0<r\leqslant h_1$,

$$\max_{0\leqslant k\leqslant d+1}\ \sup_{\delta<\rho<\infty}\frac{|(d^k/d\rho^k)(\lambda(\rho/\kappa(r)))|}{\rho^{\gamma_0-d-\varepsilon}}$$
$$\leqslant C_2\,\delta^{d+\varepsilon+N-\gamma_0}\kappa(r)^{-d-1-N}\exp(-\kappa(r)^{-\theta}\,\delta^{\theta}). \tag{6.26}$$

Observe that

$$\max_{0\leqslant k\leqslant d+1}\ \sup_{\delta<\rho<\infty}\frac{|(d^k/d\rho^k)(\lambda(\rho/\kappa(r)))|}{\rho^{\gamma_0-d-\varepsilon}}$$
$$=\max_{0\leqslant k\leqslant d+1}\kappa(r)^{-k}\sup_{\delta<\rho<\infty}\frac{|\lambda^{(k)}(\rho/\kappa(r))|}{\rho^{\gamma_0-d-\varepsilon}}$$
$$\leqslant\kappa(r)^{-d-1}\max_{0\leqslant k\leqslant d+1}\ \sup_{\delta<\rho<\infty}\frac{|\lambda^{(k)}(\rho/\kappa(r))|}{(\rho/\kappa(r))^{N}\exp(-\kappa(r)^{-\theta}\rho^{\theta})}$$
$$\times\frac{(\rho/\kappa(r))^{N}\exp(-\kappa(r)^{-\theta}\rho^{\theta})}{\rho^{\gamma_0-d-\varepsilon}}$$
$$\leqslant C_2\kappa(r)^{-d-1-N}\sup_{\delta<\rho<\infty}\rho^{d+\varepsilon+N-\gamma_0}\exp(-\kappa(r)^{-\theta}\rho^{\theta}).$$

Since $\kappa(r) \to 0$ as $r \to 0$, it is a straightforward matter to show, using elementary differential calculus, that there exists $h_1 \in (0, 1]$ such that

$$\sup_{\delta < \rho < \infty} \rho^{d+\varepsilon+N-\gamma_0} \exp(-\kappa(r)^{-\theta} \rho^{\theta})$$
$$= \delta^{d+\varepsilon+N-\gamma_0} \exp(-\kappa(r)^{-\theta} \delta^{\theta}), \qquad \forall 0 < r \leqslant h_1.$$

Hence, (6.26) holds and the claim is proved.

Therefore, by (6.20), Claim 6.21, and Claim 6.25, there exists $h_1 \in (0, 1]$ such that

$$\Gamma(r, h) \leqslant h^{\gamma_0} \operatorname{const}(d, \sigma, \delta, N, \gamma_0, \varepsilon, \eta, C_1, C_2, C_3) \times \kappa(r)^{-2d-1-N} \exp((a(d+1) h^{\theta} - 1)(\delta/\kappa(r))^{\theta}), \tag{6.27}$$

for all $0 < r \leqslant h \leqslant h_1$. Now in view of (6.18), and since δ was chosen arbitrarily in $(0, \pi)$, we may assume without loss of generality that $\delta \in (0, \pi)$ is sufficiently close to π so that

$$\limsup_{h \to 0} \kappa(h)^{\theta} \log(1/h) < \frac{(\delta - \varepsilon_1)^{\theta}}{\gamma_1}, \qquad \text{for some} \quad \varepsilon_1 > 0.$$

Hence there exists $h_2 \in (0, h_1]$ such that

$$\kappa(h) \leqslant \bar{\kappa}(h) := \left(\frac{(\delta - \varepsilon_1)^{\theta}}{\gamma_1 \log(1/h)} \right)^{1/\theta}, \qquad \forall 0 < h \leqslant h_2.$$

It can be shown, by applying elementary differential calculus to (6.27), that there exists $h_0 \in (0, h_2]$ such that

$$\sup_{0 < r \leqslant h} \Gamma(r, h) \leqslant h^{\gamma_0} \operatorname{const}(d, \sigma, \delta, N, \gamma_0, \varepsilon, \eta, C_1, C_2, C_3)$$
$$\times \bar{\kappa}(h)^{-2d-1-N} \exp((a(d+1) h^{\theta} - 1)(\delta/\bar{\kappa}(h))^{\theta}),$$

for all $0 < h \leqslant h_0$. Now, as $h \to 0$,

$$\bar{\kappa}(h)^{-2d-1-N} \exp((a(d+1) h^{\theta} - 1)(\delta/\bar{\kappa}(h))^{\theta})$$
$$= O(\bar{\kappa}(h)^{-2d-1-N} \exp(-(\delta/\bar{\kappa}(h))^{\theta}))$$
$$= O\left(\bar{\kappa}(h)^{-2d-1-N} \exp\left(-\left(\frac{\delta}{\delta - \varepsilon_1} \right)^{\theta} \gamma_1 \log(1/h) \right) \right)$$
$$= O(\exp(-\gamma_1 \log(1/h)) = O(h^{\gamma_1}).$$

Therefore,

$$\sup_{0<r\leqslant h} \Gamma(r, h) = O(h^{\gamma_0+\gamma_1}) \qquad \text{as} \quad h \to 0,$$

which, in view of (6.19), completes the proof. ∎

7. PROOF OF THEOREM 3.7

Our proof of Theorem 3.7 requires the following two lemmata.

LEMMA 7.1. *Let* $0 \leqslant a < b \leqslant \infty$ *and put* $\Omega := \{x \in \mathbb{R}^d : a < |x| < b\}$. *If* $F \in C^{d+1}(a, b)$, *then*

$$\|F(|\cdot|)\|_{W^{d+1}{}_1(\Omega)}$$

$$\leqslant \text{const}(d) \left(\int_a^b \rho^{d-1} \, |F(\rho)| \, d\rho + \max_{1 \leqslant k \leqslant \ell \leqslant d+1} \int_a^b \rho^{k-\ell+d-1} \, |F^{(k)}(\rho)| \, d\rho \right).$$

Proof. First note that $\|F(|\cdot|)\|_{L_1(\Omega)} = \text{const}(d) \int_a^b \rho^{d-1} \, |F(\rho)| \, d\rho$. For $1 \leqslant |\alpha| \leqslant d+1$ we have by Lemma 6.1 that

$$\begin{aligned}
\|D^\alpha(F(|\cdot|))\|_{L_1(\Omega)} &\leqslant \text{const}(d) \sum_{k=1}^{|\alpha|} \int_\Omega |x|^{k-|\alpha|} \, |F^{(k)}(|x|)| \, dx \\
&= \text{const}(d) \sum_{k=1}^{|\alpha|} \int_a^b \rho^{k-|\alpha|+d-1} \, |F^{(k)}(\rho)| \, d\rho \\
&\leqslant \text{const}(d) \max_{1 \leqslant k \leqslant \ell \leqslant d+1} \int_a^b \rho^{k-\ell+d-1} \, |F^{(k)}(\rho)| \, d\rho. \quad ∎
\end{aligned}$$

DEFINITION. A function F: $[0, \infty) \to \mathbb{C}$ is said to be γ *admissable* ($\gamma \in \mathbb{R}$) if $F(|\cdot|) \in C^{d+1}(\mathbb{R}^d)$ and

(i) $\sup_{0 \leqslant \rho < \infty} (1+\rho)^\gamma / |F(\rho)| < \infty$ and

(ii) $|F^{(k)}(\rho)| = O(\rho^{\gamma-k})$ as $\rho \to \infty$, $0 \leqslant k \leqslant d+1$.

The relevance of this definition to Theorem 3.7 is that the function λ is $-\gamma$ admissable while the function $1/\lambda$ is γ admissable.

LEMMA 7.2. *Let* f *be* γ *admissable and let* $\delta > 0$. *Let* $a \in (0, \infty)$ *and define* $F(\rho) := f(a\rho)$, $0 \leqslant \rho < \infty$. *The following hold*:

(1) *If* $\gamma > d$, *then* $\|F(|\cdot|)\|_{W_1^{d+1}(\delta B)} \leqslant \mathrm{const}(d, \delta, \gamma, f)(1+a)^\gamma$.

(2) *if* $\gamma < -d$ *and* $a \geqslant 1$, *then* $\|F(|\cdot|)\|_{W_1^{d+1}(\mathbb{R}^d \backslash \delta B)} \leqslant \mathrm{const}(d, \delta, \gamma, f)\, a^\gamma$.

Proof. We employ Lemma 7.1. Assume $\gamma > d$. First we have

$$\int_0^\delta \rho^{d-1}|F(\rho)|\, d\rho \leqslant \mathrm{const}(f) \int_0^\delta \rho^{d-1}(1+a\rho)^\gamma\, d\rho \leqslant \mathrm{const}(d, \delta, \gamma, f)(1+a)^\gamma.$$

Next assume that $1 = k \leqslant \ell \leqslant d+1$. Since $f(|\cdot|) \in C^{d+1}(\mathbb{R}^d)$, it follows that $F'(0) = af'(0) = 0$, and consequently we can write $F'(\rho) = \int_0^\rho F''(s)\, ds$. Hence

$$\begin{aligned}
&\int_0^\delta \rho^{k-\ell+d-1}\, |F'(\rho)|\, d\rho \\
&\quad \leqslant \mathrm{const}(d, \delta) \int_0^\delta \rho^{-1} \int_0^\rho |F''(s)|\, ds\, d\rho = \mathrm{const}(d, \delta) \int_0^\delta \log(\delta/s)\, |F''(s)|\, ds \\
&\quad \leqslant \mathrm{const}(d, \delta, f)\, a^2 \\
&\qquad \times \begin{cases} \displaystyle\int_0^\delta \log(\delta/s)(as)^{\gamma-2}\, ds & \text{if} \quad \gamma < 2 \\ \displaystyle\int_0^\delta \log(\delta/s)(1+a\delta)^{\gamma-2}\, ds & \text{else} \end{cases} \leqslant \mathrm{const}(d, \delta, \gamma, f)(1+a)^\gamma.
\end{aligned}$$

Finally, assume $2 \leqslant k \leqslant \ell \leqslant d+1$. Then

$$\begin{aligned}
&\int_0^\delta \rho^{k-\ell+d-1}\, |F^{(k)}(\rho)|\, d\rho \\
&\quad \leqslant \mathrm{const}(d, \delta)\, a^k \int_0^\delta \rho^{k-2}\, |f^{(k)}(a\rho)|\, d\rho \\
&\quad \leqslant \mathrm{const}(d, \delta, f) \begin{cases} \displaystyle a^{d+1} \int_0^\delta \rho^{d-1}(a\rho)^{\gamma-d-1}\, d\rho & \text{if} \quad d < \gamma < d+1 = k \\ \displaystyle a^k \int_0^\delta \rho^{k-2}(1+a\delta)^{\gamma-k}\, d\rho & \text{else} \end{cases} \\
&\quad \leqslant \mathrm{const}(d, \delta, \gamma, f)(1+a)^\gamma
\end{aligned}$$

which proves (1). Turning now to (2), assume that $\gamma < -d$, $a \geqslant 1$, and $0 \leqslant k \leqslant \ell \leqslant d+1$. Then

$$\int_\delta^\infty \rho^{k-\ell+d-1}\,|F^{(k)}(\rho)|\,d\rho \leqslant \text{const}(d,\delta,f)\,a^k \int_\delta^\infty \rho^{d-1}(1+a\rho)^{\gamma-k}\,d\rho$$

$$\leqslant \text{const}(d,\delta,\gamma,f)\,a^\gamma \int_\delta^\infty \rho^{d-1+\gamma-k}\,d\rho$$

$$\leqslant \text{const}(d,\delta,\gamma,f)\,a^\gamma$$

which, in view of Lemma 7.1, proves (2). ∎

Proof of Theorem 3.7. We employ Theorem 5.8 with $\gamma_0=\mu=0$ and $\varepsilon=1$. Note that $\lambda_r=(\phi(r^\theta\cdot))^\wedge = r^{-d\theta}\hat{\phi}(r^{-\theta}\cdot)=r^{-d\theta}\lambda(r^{-\theta}\,|\cdot|)$. The assumptions on ϕ ensure that λ is $-\gamma$ admissable. Since $\gamma>d$, it follows that $\hat{\phi}\in L_1$ and hence condition (i) of Theorem 5.8 holds. Define

$$\Gamma_1(r,h) := \left\| \left(\frac{\hat{\eta}(\cdot/h)}{\lambda_r}\right)^\vee \right\|_{L_1} = \left\| \left(\frac{\hat{\eta}}{\lambda_r(h\cdot)}\right)^\vee \right\|_{L_1} = r^{d\theta} \left\| \left(\frac{\hat{\eta}}{\lambda(hr^{-\theta}\,|\cdot|)}\right)^\vee \right\|_{L_1},$$

and note that by the (extended) Hausdorff–Young Theorem,

$$\Gamma_1(r,h) \leqslant r^{d\theta}\,\text{const}(d) \left\| \frac{\hat{\eta}}{\lambda(r^{-\theta}h\,|\cdot|)} \right\|_{W_1^{d+1}(\mathbb{R}^d)}$$

$$\leqslant r^{d\theta}\,\text{const}(d,\eta) \left\| \frac{1}{\lambda(r^{-\theta}h\,|\cdot|)} \right\|_{W_1^{d+1}(\delta B)},$$

where δ is the smallest positive real number such that $\operatorname{supp}\hat{\eta}\subset\delta\bar{B}$. Since λ is $-\gamma$ admissable, it follows that $1/\lambda$ is γ admissable, and hence by Lemma 7.2 (1),

$$\Gamma_1(r,h) \leqslant r^{d\theta}\,\text{const}(d,\eta,\gamma,\phi)(1+r^{-\theta}h)^\gamma.$$

Note in particular that (ii) of Theorem 5.8 now follows. Now define

$$\Gamma_2(r) := \sum_{j\in\mathbb{Z}^d} \|((1-\sigma)\,\lambda_r)^\vee\|_{L_\infty(j+Q)}$$

$$= r^{-d\theta} \sum_{j\in\mathbb{Z}^d} \|((1-\sigma)\,\lambda(r^{-\theta}|\cdot|))^\vee\|_{L_\infty(j+Q)}.$$

As was shown in the first display of the proof of Lemma 6.9,

$$\Gamma_2(r) \leqslant r^{-d\theta}\,\text{const}(d,\sigma)\,\|\lambda(r^{-\theta}\,|\cdot|)\|_{W_1^{d+1}(\mathbb{R}^d\backslash\delta' B)},$$

where δ' is the largest real for which $\operatorname{supp}(1-\sigma)\subset\mathbb{R}^d\backslash\delta' B$. Since λ is $-\gamma$ admissable and $\gamma>d$, we have by Lemma 7.2 (2) that

$$\Gamma_2(r) \leqslant r^{-d\theta}\,\text{const}(d,\sigma,\gamma,\phi)(r^{-\theta})^{-\gamma} = r^{-d\theta}\,\text{const}(d,\sigma,\gamma,\phi)\,r^{\theta\gamma}.$$

Therefore,

$$\begin{aligned}\sup_{0<r\leqslant h} \Gamma_1(r,h)\,\Gamma_2(r) &\leqslant \text{const}(d,\sigma,\eta,\gamma,\phi) \sup_{0<r\leqslant h} (1+hr^{-\theta})^{\gamma}\, r^{\theta\gamma}\\ &= \text{const}(d,\sigma,\eta,\gamma,\phi) \sup_{0<r\leqslant h} (r^{\theta}+h)^{\gamma}\\ &= \text{const}(d,\sigma,\eta,\gamma,\phi)(h^{\theta}+h)^{\gamma} = O(h^{\gamma})\end{aligned}$$

which, in view of Theorem 5.8, completes the proof. ▮

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REFERENCES

1. M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions," Dover, New York, 1970.
2. R. K. Beatson and W. A. Light, Quasi-interpolation in the absence of polynomial reproduction, *in* "Numerical Methods of Approximation Theory" (D. Braess and L. L. Schumaker, Eds.), pp. 21–39, Birkhäuser, Basel, 1992.
3. C. Bennet and R. Sharpley, "Interpolation of Operators," Academic Press, San Diego, 1988.
4. C. de Boor, R. A. DeVore, and A. Ron, Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$, *Trans. Amer. Math. Soc.* **341** (1994), 787–806.
5. C. de Boor and A. Ron, Fourier analysis of approximation orders from principal shift-invariant spaces, *Constr. Approx.* **8** (1992), 427–462.
6. M. D. Buhmann, Multivariate interpolation in odd-dimensional Euclidean spaces using multiquadrics, *Constr. Approx.* **6** (1990), 21–34.
7. M. D. Buhmann, On quasi-interpolation with radial basis functions, *J. Approx. Theory* **72** (1993), 103–130.
8. M. D. Buhmann, New developments in the theory of radial basis function interpolation, *in* "Multivariate Approximation: From CAGD to Wavelets" (K. Jetter and F. I. Utreras, Eds.), pp. 35–75, World Scientific, Singapore, 1993.
9. M. D. Buhmann and N. Dyn, Spectral convergence of multiquadric interpolation, *Proc. Edinburgh Math. Soc.* **36** (1993), 319–333.
10. M. D. Buhmann, N. Dyn, and D. Levin, On quasi-interpolation by radial basis functions with scattered centres, *Constr. Approx.* **11** (1995), 239–254.
11. M. D. Buhmann and A. Ron, Radial basis functions: L_p-approximation orders with scattered centres, *in* "Wavelets, Images, and Surface Fitting, Chamonix-Mont-Blanc, 1993," pp. 93–112, A. K. Peters, Wellesley, MA, 1994.

12. N. Dyn, Interpolation and approximation by radial and related functions, *in* "Approximation Theory VI" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 211–234, Academic Press, New York, 1989.
13. N. Dyn, I. R. H. Jackson, D. Levin, and A. Ron, On multivariate approximation by integer translates of a basis function, *Israel J. Math.* **78** (1992), 95–130.
14. N. Dyn and A. Ron, Radial basis function approximation: From gridded centres to scattered centres, *Proc. London Math. Soc. (3)* **71** (1995), 76–108.
15. H. G. Feichtinger and K. Gröchenig, Irregular sampling theorems and series expansions of band-limited functions, *J. Math. Anal. Appl.* **167** (1992), 530–556.
16. I. M. Gelfand and G. E. Shilov, "Generalized Functions," Vol. 1, Academic Press, San Diego, 1964.
17. I. R. H. Jackson, An order of convergence for some radial basis functions, *IMA J. Numer. Anal.* **9** (1989), 567–587.
18. M. J. Johnson, An upper bound on the approximation power of principal shift-invariant spaces, *Constr. Approx.* **13** (1997), 155–176.
19. M. J. Johnson, On the approximation power of principal shift-invariant subspaces of $L_p(\mathbb{R}^d)$, *J. Approx. Theory* **91** (1997), 279–319.
20. W. R. Madych and S. A. Nelson, Bounds on multivariate polynomials and exponential error estimates for multiquadric interpolation, *J. Approx. Theory* **70** (1992), 94–114.
21. J. Peetre, New thoughts on Besov spaces, Math. Dept., Duke University, Durham, NC, 1976.
22. M. J. D. Powell, The theory of radial basis function approximation in 1990, *in* "Advances in Numerical Analysis. II. Wavelets, Subdivision, and Radial Functions" (W. A. Light, Ed.), pp. 105–210, Oxford Univ. Press, Oxford, 1992.
23. A. Ron, The L_2-approximation orders of principal shift-invariant spaces generated by a radial basis function, *in* "Numerical Methods of Approximation Theory" (D. Braess and L. L. Schumaker, Eds.), Vol. 9, pp. 245–268, Birkhäuser, Basel, 1991.
24. R. Schaback, Approximation by radial basis functions with finitely many centers, *Constr. Approx.* **12** (1996), 331–340.
25. H. Wendland, Error estimates for interpolation by compactly supported radial basis functions of minimal degree, *J. Approx. Theory* **93** (1998), 258–272.
26. Z. Wu and R. Schaback, Local error estimates for radial basis function interpolation of scattered data, *IMA J. Numer. Anal.* **13** (1993), 13–27.